Some clarification about the transversality condition

The general transversality condition:

$$\left(\nabla_{\underline{x}}\phi + (\nabla_{\underline{x}}\underline{\Psi})^T\underline{\nu} - \underline{\lambda}\right)^T\Big|_{t=T} d\underline{x}(T) + \left(\frac{\partial\phi}{\partial t} + \left(\frac{\partial\underline{\Psi}}{\partial t}\right)^T\underline{\nu} + H\right)\Big|_{t=T} dT = 0$$

where:

$\phi \equiv$ Terminal cost (Scalar f^{n}), $\underline{\Psi} \equiv$ Terminal constraint (vector f^{n})

$\underline{x}(t) \equiv$ state vector, $\underline{\lambda}(t) \equiv$ costate vector

$\underline{\nu} \equiv$ Lagrange multiplier vector, $H \equiv$ Hamitonian

- The <u>transversality condition</u> comes from the PMP proof which we will NOT do in class

- The intuitive reason why there is a sum of the form:

$$\left.(\ldots)\right|_{t=T} d\underline{x}(T) + \left.(\ldots)\right|_{t=T} dT = 0,$$

is that the variations $d\underline{x}(T)$ and dT may or may not be independent, depending on how you formulate the OCP.

- Several cases to consider:

$\boxed{\text{Case I}}$ If final time (T) is fixed but final state $\underline{x}(T)$ is free, then $dT = 0$, but $d\underline{x}(T) \neq 0$

$\therefore$ coeff. of $d\underline{x}(T) = 0$.

$\boxed{\text{Case II}}$ If final state ($\underline{x}(T)$) is fixed but final time (T) is free (e.g. Brachistochrone)

then:

$d\underline{x}(T) = 0$, $dT \neq 0 \Leftrightarrow$ coeff. of $dT = 0$.

$\boxed{\text{Case III}}$ If BOTH $\underline{x}(T)$ and T are fixed,

then $d\underline{x}(T) = dT = 0 \Leftrightarrow 0 + 0 = 0$

(Transversality condⁿ gives no extra info)

(e.g. min. energy state transfer)

$\boxed{\text{Case IV}}$ If BOTH $\underline{x}(T)$ and T are free

(e.g. minimum time missile intercept), then

$d\underline{x}(T) \neq 0$, $dT \neq 0$.

Now we need to consider 2 sub-cases.

$\boxed{\text{Case IV. A}}$ The variations $d\underline{x}(T)$ and dT are independent.

(Contd.)

Then, coeff. of $d\underline{x}(T) = 0$
AND coeff. of $dT = 0$

$$\Leftrightarrow \left(\nabla_{\underline{x}} \phi + (\nabla_{\underline{x}} \underline{\psi})^T \underline{\nu} - \underline{\lambda} \right) \Big|_{t=T} = 0$$

AND

$$\left(\frac{\partial \phi}{\partial t} + \left(\frac{\partial \underline{\psi}}{\partial t}\right)^T \underline{\nu} - H \right) \Big|_{t=T} = 0$$

Notice here that the terminal constraint

$$\underline{\psi}\,(\underline{x}(T), T) = 0$$

may involve T. Once $\underline{\psi}(\underline{x}(T), T)$ is modeled, there is NO extra dependence of $d\underline{x}(T)$ on T. This is why in the min. time intercept problem, $d\underline{x}(T)$ is indep. of dT.

$\boxed{\text{Case IV. B}}$ $d\underline{x}(T) \neq 0$, $dT \neq 0$,

Furthermore, $d\underline{x}(T)$ and dT are dependent.

e.g. If we demand that $\underline{x}(T)$ needs to lie on a moving pt. $\underline{p}(t)$, otherwise $\underline{x}(T)$ and T are free

Then $\underline{x}(T) = \underline{p}(T)$

and $d\underline{x}(T) = \dfrac{d\underline{p}(T)}{dT} dT$. ---- $(*)$

Now set $\underline{\Psi} \equiv \underline{0}$ (zero function) in the transversality condition and substitute $(*)$ for writing $d\underline{x}(T)$ as function of dT:

$$\left(\nabla_{\underline{x}} \phi - \underline{\lambda}\right)^T \Big|_{t=T} \frac{d\underline{p}(T)}{dT} dT + \left(\frac{\partial \phi}{\partial t} + H\right)\Big|_{t=T} dT = 0$$

Since $dT \neq 0$, the above gives (next pg.)

$$\left(\nabla_{\underline{x}}\phi - \underline{\lambda}\right)^T\Big|_{t=T} \frac{d\underline{p}(T)}{dT} + \left(\frac{\partial\phi}{\partial t} + H\right)\Big|_{t=T} = 0. \quad \text{---} (**)$$

One may formulate the <u>same OCP</u> in the $\boxed{\text{Case IV.A}}$ form, or in the $\boxed{\text{Case IV.B}}$ form, depending on how the modeling is done.