Lec. 20 (Last!!!)

Completely observable case (i.e., $y \equiv x$)

Deterministic HJB

Problem: $\min_{\gamma(\cdot) \in \Gamma} \phi(\underline{x}(T), T) + \int_0^T L(t, \underline{x}, u)\, dt$

s.t. $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$, $\quad \underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$, $\underline{u} \in \mathcal{U} \subseteq \mathbb{R}^m$

Find $u^*(\underline{x}, t) = \gamma^*(\underline{x}, t)$

PDE for the value f^n $V(\underline{x}, t)$ is called HJB PDE

$$\frac{\partial V}{\partial t} + \min_{\underline{u} \in \mathcal{U}} \left\{ L(t, \underline{x}, \underline{u}) + \left\langle \frac{\partial V}{\partial \underline{x}}, \underline{f}(t, \underline{x}, \underline{u}) \right\rangle \right\} = 0, \quad \text{s.t. } V(\underline{x}(T), T) = \phi(\underline{x}(T), T)$$

$$\Updownarrow$$

$$\frac{\partial V}{\partial t} + H_{opt}\left(t, \underline{x}, \frac{\partial V}{\partial \underline{x}}\right) = 0 \quad \text{s.t. (copy)}$$

1st order PDE, nonlinear PDE

Terminal value problem $\Leftrightarrow$ IVP by $\tau := T - t$

$\Updownarrow$

IVP:

$$\frac{\partial V}{\partial \tau} + H_{opt}\left(\tau, \underline{x}, \frac{\partial V}{\partial \underline{x}}\right) = 0$$

s.t. $V(\tau = 0, \underline{x}) = \phi(\underline{x})$

How to Solve: Numerical sol^n: Method of characteristics

Analytical: e.g. LQR: $V(t, \underline{x}) \equiv \frac{1}{2} \underline{x}^T P(t)\, \underline{x}$ (guess)

Derive Riccati ODE etc.
Kalman gain

↪ **Moral:** Guess can help solve this PDE.

Analytical: (no guess)

Suppose OCP is time invariant.

Suppose also that H_{opt} is indep. of $\underline{x}$.

(i.e.) $H_{opt}(\cdot) \equiv H_{opt}\left(\frac{\partial V}{\partial \underline{x}}\right)$

AND $H_{opt}(\underline{z})$ is cnvx in $\underline{z}$

AND $H_{opt}(\underline{z})$ is 1-coercive, (i.e.) $\lim_{\|\underline{z}\| \to \infty} \frac{H(\underline{z})}{\|\underline{z}\|} = +\infty$

(superlinear)

Then we can give semi-analytical variational formula for $V(\tau, \underline{x})$.

Hopf–Lax Representation Formula:

$$V(\tau, \underline{x}) = \min_{\underline{y} \in \mathbb{R}^n} \left\{ \phi(\underline{y}) + \underbrace{\tau H^*_{opt}\left(\frac{\underline{x}-\underline{y}}{\tau}\right)}_{} \right\}$$

$$H^*(\underline{a}) := \underbrace{\max_{\underline{z} \in \mathbb{R}^n} \left\{ \underline{a}^T \underline{z} - H(\underline{z}) \right\}}_{\text{Legendre-Fenchel conjugate or Convex conjugate of } H(\cdot)}$$

$\underbrace{H^*(\underline{a})}_{\equiv \text{ Lagrangian } (L)}$

<u>Example:</u> If <u>HJB PDE</u>: $\frac{\partial V}{\partial t} + \frac{1}{2}\left\| \frac{\partial V}{\partial \underline{x}} \right\|_2^2 = 0$

$$H(\underline{z}) = \frac{1}{2}\underline{z}^T\underline{z}$$

Then $H^*(\underline{a}) = \frac{1}{2}\underline{a}^T\underline{a} \Rightarrow \therefore \tau H^*_{opt}\left(\frac{\underline{x}-\underline{y}}{\tau}\right) = \frac{1}{2\tau}\|\underline{x}-\underline{y}\|_2^2$

$\therefore$ By Hopf-Lax :

$$V(\underline{x}, \tau) = \min_{\underline{y} \in \mathbb{R}^n} \left\{ \phi(\underline{y}) + \frac{1}{2\tau} \| \underline{x} - \underline{y} \|_2^2 \right\}$$

$$= \frac{1}{\tau} \min_{\underline{y} \in \mathbb{R}^n} \left\{ \frac{1}{2} \| \underline{x} - \underline{y} \|_2^2 + \tau \phi(\underline{y}) \right\}$$

$$= \frac{1}{\tau} \underbrace{\mathrm{prox}^{\|\cdot\|_2}_{\tau\phi(\cdot)} (\underline{x})}_{\text{Moreau-Yosida proximal operator}}$$

Stochastic HJB (still completely observed):

$$\boxed{\dot{\underline{x}} = f(t,\underline{x},\underline{u}) + g(t,\underline{x},u) \times \text{noise}} \checkmark$$

$$\Leftrightarrow \underbrace{d\underline{x}}_{n\times 1} = \underbrace{f(t,\underline{x},u)}_{n\times 1}\, dt + \underbrace{g(t,\underline{x},u)}_{n\times p}\, \overbrace{d\underline{w}}^{p\times 1}$$

(Stochastic differential eq.

drift coeff.

Diffusion coeff.

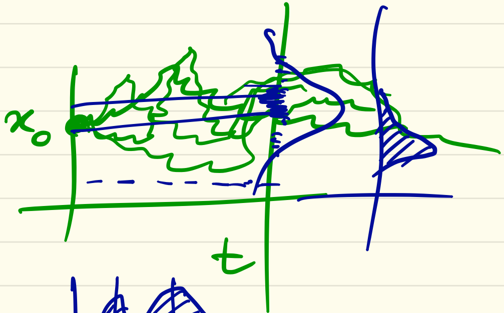
process noise $d\underline{w} \sim \mathcal{N}(\underline{0}, I\,dt)$

Objective

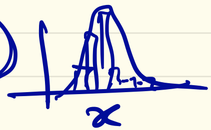
$$\min_{\gamma \in \Gamma(\cdot)} \mathbb{E}\left[\phi(\underline{x}(T), T) + \int_0^T L(\underline{x}, u, t)\, dt\right]$$

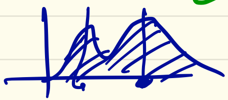
s.t. this dynamics

$$\text{Find } u^* = \gamma^*(\underline{x}, t)$$



Fixed t





Stochastic HJB:

(Remember: Here, we have no PMP)

Define Hamiltonian:

$$H\left(\underline{x}, \underline{u}, t, V, \underbrace{\frac{\partial V}{\partial \underline{x}}}_{W}, \underbrace{\frac{\partial^2 V}{\partial \underline{x}^2}}_{Hess(V)}\right)$$

$$:= L(t, \underline{x}, \underline{u}) + \left\langle \frac{\partial V}{\partial \underline{x}}, f(t, \underline{x}, \underline{u}) \right\rangle +$$

$$\underbrace{\frac{1}{2} tr\left(\underbrace{g^T(t, \underline{x}, \underline{u})}_{p \times n} \underbrace{Hess(V)}_{n \times n} \underbrace{g(t, \underline{x}, \underline{u})}_{n \times p} \right)}_{= \frac{1}{2} tr\left(\underbrace{g g^T}_{n \times n} Hess(V) \right)}$$

Diffusion tensor

Stoc. HJB :

$$\frac{\partial V}{\partial t} + \min_{u \in \mathcal{U}} H\left(t, \underline{x}, \underline{u}, V, \frac{\partial V}{\partial \underline{x}}, \frac{\partial^2 V}{\partial \underline{x}^2}\right) = 0$$

$$\Leftrightarrow \frac{\partial V}{\partial t} + H_{opt}\left(t, \underline{x}, V, \frac{\partial V}{\partial \underline{x}}, \frac{\partial^2 V}{\partial \underline{x}^2}\right) = 0,$$

2nd order nonlinear PDE

$$V(\underline{x}(T), T) = \phi(\underline{x})$$

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side remark:

$$d\underline{x} = \underline{f}(\underline{x}, t)\, dt + \underline{g}(\underline{x}, t)\, dw$$

$y : \mathbb{R}^n \mapsto \mathbb{R}$ $\quad y = y(\underline{x}, t) \leftarrow$ Deterministic function

Itô's Lemma :

sub.

$$dy = \frac{\partial y}{\partial t} dt + (\nabla y)^T d\underline{x} + \frac{1}{2} (d\underline{x})^T (\mathrm{Hess}(\underline{y}))^T d\underline{x}$$

$$= \left\{ \frac{\partial y}{\partial t} + (\nabla y)^T \underline{f}(\underline{x}, t) + \frac{1}{2} \mathrm{tr}\left(g^T \mathrm{Hess}(y)\, g\right) \right\} dt + (\nabla y)^T g(\underline{x}, t)\, d\underline{w}$$

Fokker-Plack or Kolmogorov Forward PDE:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho f) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(gg^T\rho)$$

$$\rho(x, t=0) = \rho_0(x)$$

$$\Leftrightarrow \frac{\partial \rho}{\partial t} = \mathcal{L}_{\text{Fokker-Plack}}\,\rho$$

$$\text{or } \mathcal{L}_{\text{Forward Kolmogorov}}\,\rho$$

Backward Kolmogorov PDE

$$\frac{\partial p}{\partial s} = \mathcal{L}_{\substack{\text{Backward}\\ \text{Kolmogorov}}}\, p,$$

$$= \left\langle \frac{\partial p}{\partial x}, f \right\rangle + \frac{1}{2}\operatorname{tr}(gg^T \operatorname{Hess}(p))$$

Formally,

$$\mathcal{L}_{\substack{\text{Backward}\\ \text{Kol.}}} = (\mathcal{L}_{\text{Forward}})^{*}.$$

adjoint

$$\langle \mathcal{L}\rho, v\rangle = \langle \rho, \mathcal{L}^{*} v\rangle$$

$$\forall\ \rho, v \in L^2$$

Stochastic HJB:

$$\frac{\partial V}{\partial t} + \min_{u \in \mathcal{U}} \left\{ L(t, \underline{x}, u) + \mathcal{L}_{\substack{\text{Backward} \\ \text{Kolmogorov}}} V \right\} = 0$$