Lecture # 19

State estimation for Linear Gaussian System

No control yet

$$\begin{cases} \underline{x}_{k+1} = A\underline{x}_k + G\underline{w}_k \leftarrow \text{Process noise} \sim \mathcal{N}(\underline{0}, Q) \\ \underline{y}_k = C\underline{x}_k + H\underline{v}_k \leftarrow \text{Measurement noise} \sim \mathcal{N}(\underline{0}, R) \end{cases}$$

Also known as, Gauss-Markov system

Initial condition $\underline{x}_0 \sim \mathcal{N}(\underline{\bar{x}}_0, \Sigma_0)$, $\Sigma_0 \succeq 0$, $\underbrace{Q \succeq 0,\ R \succ 0}$,

Assume: Process & Measurement noise are indep., and they are also indep. for diff. values of k.

We know: $p(\underline{x}_k \mid y^k) = \mathcal{N}(\underbrace{\hat{\underline{x}}_{k|k}}_{\text{Random vector}}, \underbrace{\Sigma_{k|k}}_{\text{Deterministic Matrix}})$

Measurement updated PDF

$$p(\underline{x}_{k+1} \mid y^k) = \mathcal{N}\left(\hat{\underline{x}}_{k+1|k}, \Sigma_{k+1|k}\right) \leftarrow \text{Time updated PDF}$$

All we need, is a recursive scheme:

$$\underbrace{\begin{matrix}\hat{\underline{x}}_{k|k} \longrightarrow \hat{\underline{x}}_{k+1|k} \\ \Sigma_{k|k} \longrightarrow \Sigma_{k+1|k}\end{matrix}}_{\text{Time update}} \quad \underbrace{\begin{matrix}\longrightarrow \hat{\underline{x}}_{k+1|k+1} \\ \longrightarrow \Sigma_{k+1|k+1}\end{matrix}}_{\text{Measurement update}}$$

Time update: $\hat{\underline{x}}_{k+1|k} := \mathbb{E}[\underline{x}_{k+1} \mid y^k]$

conditional mean

$$= \mathbb{E}[A\underline{x}_k + G\underline{w}_k \mid y^k]$$

$$= \underbrace{\mathbb{E}[A\underline{x}_k \mid y^k]}_{} + \underbrace{\mathbb{E}[G\underline{w}_k \mid y^k]}_{}$$

$$= A\hat{\underline{x}}_{k|k} + 0$$

$$\Rightarrow \boxed{\hat{\underline{x}}_{k+1|k} = A\hat{\underline{x}}_{k|k}}, \quad \underbrace{\hat{\underline{x}}_{0|-1} := \bar{\underline{x}}_0}_{\text{Initial condition}}$$

initial mean

Covariance

$$\Sigma_{k+1|k} := \mathbb{E}\left[\left(\underline{x}_{k+1} - \underline{\hat{x}}_{k+1|k}\right)\left(\underline{x}_{k+1} - \underline{\hat{x}}_{k+1|k}\right)^T\right]$$

$$= \mathbb{E}\left[\left(A\underline{x}_k + G\underline{w}_k - A\hat{x}_{k|k}\right)\left(A\underline{x}_k + G\underline{w}_k - A\hat{x}_{k|k}\right)^T\right]$$

$$= \mathbb{E}\left[\left(A\left(\underline{x}_k - \hat{x}_{k|k}\right) + G\underline{w}_k\right)\left(A\left(\underline{x}_k - \hat{x}_{k|k}\right) + G\underline{w}_k\right)^T\right]$$

$$\Rightarrow \boxed{\Sigma_{k+1|k} = A\Sigma_{k|k}A^T + GQG^T, \quad \Sigma_{0|-1} = \Sigma_0}$$

Measurement update:

$$\underline{\hat{y}}_{k+1|k} := \mathbb{E}\left[\underline{y}_{k+1} \mid \underline{y}_k\right] \quad (\text{by def}^{=})$$

$$= \mathbb{E}\left[C\underline{x}_{k+1} + H\underline{v}_{k+1} \mid \underline{y}_k\right]$$

Now let

$$\underbrace{\underline{\tilde{y}}_{k+1|k}}_{\text{Error/ Innovation process}} := \underline{y}_{k+1} - \underline{\hat{y}}_{k+1|k}$$

$$= C\underline{x}_{k+1} + H\underline{v}_{k+1} - C\underline{\hat{x}}_{k+1|k}$$

Likewise, define state estimation error: $\tilde{x}_{k+1|k}$

$$\tilde{x}_{k+1|k} := x_{k+1} - \hat{x}_{k+1|k}$$

Then from prev. page bottom,

$$\tilde{\underline{y}}_{k+1|k} = C\,\tilde{\underline{x}}_{k+1|k} + H\underline{v}_{k+1} \quad ---(*)$$

$$\therefore \Sigma_{\tilde{y}_{k+1|k},\, \tilde{y}_{k+1,k}} = C\,\Sigma_{k+1|k}\,C^T + HRH^T$$

Now,

$$\hat{x}_{k+1|k+1} = \mathbb{E}\left[\underline{x}_{k+1} \mid y^{k+1}\right]$$

$$= \mathbb{E}\left[\underline{x}_{k+1} \mid y^{k}, \tilde{\underline{y}}_{k+1|k}\right]$$

$$= \hat{\underline{x}}_{k+1|k} + \Sigma_{\underline{x}_{k+1},\, \tilde{y}_{k+1,k}}\, \Sigma^{-1}_{\tilde{\underline{y}}_{k+1|k},\, \tilde{\underline{y}}_{k+1,k}} \left(\underline{y}_{k+1} - C\hat{\underline{x}}_{k+1|k}\right)$$

This is like $\mathbb{E}[a|b]$ when $\begin{pmatrix} a \\ b \end{pmatrix}$ is joint Gaussian

$$= \hat{\underline{x}}_{k+1|k} + \underbrace{\Sigma_{\underline{x}_{k+1|k}\,\tilde{\underline{y}}_{k+1|k}}} \left(C \Sigma_{k+1|k} C^T + HRH^T \right)^{-1} \left(\underline{y}_{k+1} - C \hat{\underline{x}}_{k+1|k} \right)$$

Now,

$$\Sigma_{\underline{x}_{k+1|k}\,\tilde{\underline{y}}_{k+1|k}} = \mathbb{E}\left[\underline{x}_{k+1|k} \left(C \tilde{\underline{x}}_{k+1|k} + H v_{k+1} \right)^T \right]$$

(from (*))

$$= \mathbb{E}\left[\left(\hat{\underline{x}}_{k+1|k} + \tilde{\underline{x}}_{k+1|k} \right) \left(C \tilde{\underline{x}}_{k+1|k} + H \underline{v}_{k+1} \right)^T \right]$$

$$= \Sigma_{k+1|k} C^T$$

Therefore,

$$\boxed{\hat{\underline{x}}_{k+1|k+1} = \hat{\underline{x}}_{k+1|k} + \Sigma_{k+1|k} C^T \left(C \Sigma_{k+1|k} C^T + HRH^T \right)^{-1} \left(\underline{y}_{k+1} - C \hat{\underline{x}}_{k+1|k} \right)}$$

Likewise,

$$\Sigma_{k+1|k+1} = \Sigma_{k+1|k} - \Sigma_{k+1|k} C^T \left(C \Sigma_{k+1|k} C^T + HRH^T \right)^{-1} C \Sigma_{k+1|k} .$$

Kalman Filter (in discrete time)

$$\underline{x}_0 \sim \mathcal{N}(\bar{\underline{x}}_0, \Sigma_0)$$

System:

$$\underline{x}_{k+1} = A \underline{x}_k + B \underline{u}_k^g + G \underline{w}_k ,$$

$$\underline{y}_k = C \underline{x}_k + H \underline{v}_k .$$

$\underline{w}_k \sim \mathcal{N}(0, Q)$; $\underline{v}_k \sim \mathcal{N}(0, R)$; $\Sigma_0, Q \succcurlyeq 0$, $R \succ 0$

Kalman filter:

$$\hat{\underline{x}}_{0|-1} = \bar{\underline{x}}_0 .$$

$$\hat{\underline{x}}_{k+1|k} = A \hat{\underline{x}}_{k|k} + B u_k^g ,$$

$$\hat{\underline{x}}_{k+1|k+1} = \hat{\underline{x}}_{k+1|k} + \Sigma_{k+1|k} C^T \left(C \Sigma_{k+1|k} C^T + HRH^T \right)^{-1} \left(\underline{y}_{k+1} - C \hat{\underline{x}}_{k+1|k} \right)$$

$$\Sigma_{k+1|k} = A\Sigma_{k|k}A^T + GQG^T, \quad \Sigma_{0|-1} = \Sigma_0$$

$$\Sigma_{k+1|k+1} = \Sigma_{k+1|k} - \Sigma_{k+1|k}C^T(C\Sigma_{k+1|k}C^T - HRH^T)^{-1}C\Sigma_{k+1|k}$$

This completes the <u>Kalman filter</u>.

② Combining time & Measurement updates:

$$p(\underline{\hat{x}}_{k+1|k+1} \mid \hat{x}_{k|k}, u_k^*)$$

$$= \mathcal{N}\Big(A\hat{x}_{k|k} + Bu_k^*, \underbrace{\Sigma_{k+1|k} - \Sigma_{k+1|k+1}}_{=:\Delta_{k+1}}\Big)$$

Linear Gaussian System with Quadratic Cost:

(LQG Problem)

$$\min \mathbb{E}\left[\underline{x}_N^T P_N \underline{x}_N + \sum_{k=0}^{N-1}\left(\underline{x}_k^T P_k \underline{x}_k + \underline{u}_k^T T_k \underline{u}_k\right)\right],$$

$P_k \succcurlyeq 0$, $T_k \succ 0$ $\forall k$

DP eqⁿ: (Backward Recursion)

$$V_N(\hat{\underline{x}}) = \hat{\underline{x}}^T P_N \hat{\underline{x}} + \mathrm{tr}\left(P_N \Sigma_{N|N}\right)$$

$\Rightarrow \mathbb{E}\left[\underline{x}_N^T P_N \underline{x}_N \mid y^N\right]$

$= \mathbb{E}\left[\underline{x}_N^T P_N \underline{x}_N \mid \right.$

$\left. \underline{x}_N \sim \mathcal{N}\left(\hat{x}_{N|N}, \Sigma_{N|N}\right)\right]$

$$V_k(\hat{\underline{x}}) = \min_{\underline{u} \in \mathcal{U}} \Big[\hat{\underline{x}}^T P_k \hat{\underline{x}} + \mathrm{tr}\left(P_k \Sigma_{k|k}\right) + \underline{u}^T T_k \underline{u} + \mathbb{E}\left[V_{k+1}(\underline{z}) \mid z \sim \mathcal{N}\left(A\hat{x} + Bu, \Delta_{k+1}\right)\right]\Big]$$

Claim: The value f^n in DP recursion has the form:

$$V_K(\underline{\hat{x}}) = \underbrace{\underline{\hat{x}}\, S_K\, \underline{\hat{x}}}_{} + \delta_K$$

Proof: Clearly, true for $K=N$. (Backward Induction)

Assume, it's true for $(K+1)^{th}$ stage. Let's prove it's still true for K^{th} stage.

Digression: $(*) = \min_u \left[u^T J u + u^T K x + x^T K^T u \right]$, $J \succ 0$.

one way

Differentiate, and equate to zero

$u = -J^{-1} K x \leftarrow$ argmin

$\Rightarrow \therefore$ min value $= -x^T K^T J^{-1} x$.

another way

Completion of squares, next.

$$(*) = \min_{u} \Big[(u + J^{-1}Kx)^T J \,(u + J^{-1}Kx) - x^T K^T J^{-1} x \Big]$$

$$\therefore \ \text{argmin} = -J^{-1}Kx$$

$$\therefore \ \text{min. value} = -x^T K^T J^{-1} x$$

Back to proof:

$$V_k(\underline{\hat{x}}) = \min_{u \in \mathcal{U}} \Big[\underline{\hat{x}}^T P_k \underline{\hat{x}} + \operatorname{tr}(P_k \Sigma_{k|k}) + u^T T_k u + (A\underline{\hat{x}} + B\underline{u})^T S_{k+1} (A\hat{x} + Bu) + \operatorname{tr}(S_{k+1} \Delta_{k+1}) + \delta_{k+1} \Big]$$

(contd.)

$$= \min_{u} \Big[u^T (T_k + B^T S_{k+1} B) u + u^T B^T S_{k+1} A \hat{x} + \hat{x}^T A^T S_{k+1} B u + \hat{x}^T P_k \hat{x} + \hat{x}^T A^T S_{k+1} A \hat{x} + \mathrm{tr}(P_k \Sigma_{k|k}) + \mathrm{tr}(S_{k+1} \Delta_{k+1}) + s_{k+1} \Big]$$

(Extract quadratic form visually)

$$= - \hat{x}^T A^T S_{k+1} B (T_k + B^T S_{k+1} B)^{-1} B^T S_{k+1} A \hat{x} + \hat{x}^T P_k \hat{x} + \hat{x} A^T S_{k+1} A \hat{x} + \mathrm{tr}(P_k \Sigma_{k|k}) + \mathrm{tr}(S_{k+1} \Delta_{k+1}) + s_{k+1}$$

Then

$$\boxed{V_k(\hat{\underline{x}}) = \hat{x}^T S_k \hat{x} + s_k, \quad \text{where}}$$

$$S_k = -A^T S_{k+1} B (T_k + B^T S_{k+1} B)^{-1} B^T S_{k+1} A$$

$$+ P_k + A^T S_{k+1} A$$

$$s_k = tr(P \Sigma_{k|k}) + tr(S_{k+1} \Delta_{k+1}) + s_{k+1}$$

<u>Bndy cond^ns</u>

$$S_N = P_N$$

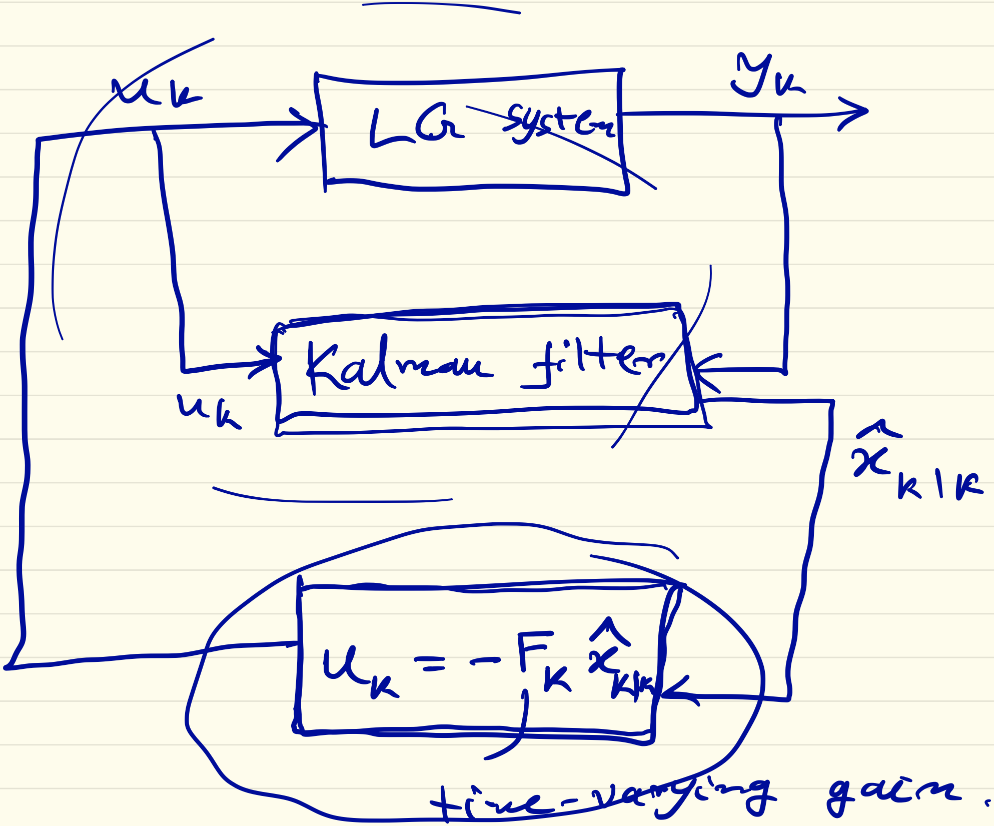
$$s_N = tr(P_N \Sigma_{N|N}).$$

$\therefore$ Optimal Control:

$$\underline{u}_k(\hat{x}_{k|k}) = -\underbrace{\underbrace{\boxed{\left(T_k + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A}}_{F_k}\hat{x}_{k|k}}_{\text{linear feedback with time-varying gain.}}$$

$\therefore$ feedback gain is deterministic
($\therefore$ can be pre-computed offline & stored as a look-up table)

architecture :



Observation:

F_k depends on cost f^{n} matrices P_i, T_i, and system matrices A, B

(i.e.) $F = fn(A, B, P, T)$

But does not depend on $\underset{\sim}{Q}$, $\underset{\sim}{R}$, $\underset{\sim}{\Sigma_0}$, $\underset{\sim}{\Sigma_{k|k}}$, $\underline{\underline{H}}$, $\underline{\underline{G}}$.

$\therefore$ Feedback gain remains invariant if we set $Q, R, \Sigma_0, \Sigma_{k|k}, H, G \equiv 0$

(i.e.) feedback gain is same as that of the deterministic LQ problem.

(i.e.) $x_{k+1} = Ax_k + Bu_k$

$\min \left[x_N^T P_N x_N + \sum_{k=0}^{N-1} (x_k^T P_k x_k + u_k^T T_k u_k) \right]$

This property is called "Certainty Equivalence Principle".

Then what does the noise do: It drives the cost.

Partially Observed Systems:

- General result: Separated Policy

$$u_k^* = \gamma_k^*(p_{k|k})$$

- Linear Gaussian with General Cost:

$$u_k^* = \gamma_k^*(\hat{x}_{k|k})$$ (still separated, but much easier to implement)

- Linear Gaussian with Quadratic Cost:

separation principle + certainty equivalence

(i.e.)

① $$x_{k+1} = Ax_k + Bu_k$$
$$y_k = x_k$$
$\underbrace{\hphantom{xxxxxxxxxxxxxxx}}$ completely observed linear deterministic

② $$x_{k+1} = Ax_k + Bu_k + Gw_k$$
$$y_k = x_k$$
$\underbrace{\hphantom{xxxxxxxxxxxxxxx}}$ completely observed linear stochastic

③ $$x_{k+1} = Ax_k + Bu_k + Gw_k$$
$$y_k = Cx_k + Hv_k$$
$\underbrace{\hphantom{xxxxxxxxxxxxxxx}}$ Partially observed linear Stochastic

Above ①, ②, ③ has same $u^*(\cdot)$

in ① : $u_k^* = -F_k x_k$ (LQR)

in ② : $u_k^* = -F_k x_k$ (LQR + process noise) in ③ : $u_k^* = -F_k \hat{x}_{k|k}$ (LQG).

Linear Non-Gaussian Systems:

$$\mathbb{E}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\Sigma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$$

Then $\hat{x} := \bar{x} + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \bar{y})$

$$\tilde{x} = x - \hat{x}$$

Then $\tilde{x}$ & y are uncorrelated.

Claim: Let Fy be any linear estimate of x, based on y. Minimum Mean Square Error

Then the best (in MMSE sense) linear estimate is $\hat{x} = \bar{x} + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \bar{y})$

Proof: $\Sigma_{(x-Fy)(x-Fy)}$

$$= \Sigma_{(x-\hat{x}+\hat{x}-Fy)(x-\hat{x}+\hat{x}-Fy)}$$

($\because$ Uncorrelated, just showed)

$$= \Sigma_{(x-\hat{x})(x-\hat{x})} + \Sigma_{(\hat{x}-Fy)(\hat{x}-Fy)}$$

$$\geq \Sigma_{(x-\hat{x})(x-\hat{x})}$$

$\therefore$ Kalman Filter is the best linear estimator even for non-Gaussian.

Dynamic Programming in Continuous Time

Deterministic DP : Setting:

$\boxed{u = \gamma(t, x)}$

$$\min_{\underline{\gamma}(t,\underline{x}) \in \Gamma} J \quad \text{cost function}$$

$$\text{s.t.} \quad \underbrace{\underline{\dot{x}} = \underline{f}(t, \underline{x}, \underline{u})}_{\text{Deterministic controlled dynamics}}$$

$\underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n, \quad \underline{u} \in \mathcal{U} \subseteq \mathbb{R}^m$

$$J = \underbrace{\phi(\underline{x}(T))}_{\text{terminal cost}} + \int_0^T \underbrace{L(t, x, u)}_{\text{Lagrangian}}\, dt$$

Define: Value f^{n} in continuous time:

$$V(t, \underline{x}) := \inf_{\substack{\gamma(s,x) \in \Gamma \\ t \leq s \leq T}} J$$

$$V(T, \underline{x}) = \phi(x(T), T)$$

Wanted: Dynamic Programming Equation for the dependent variable V, function

indep. variables: $(t, \underline{x}) \in (0, T] \times \mathcal{X}$

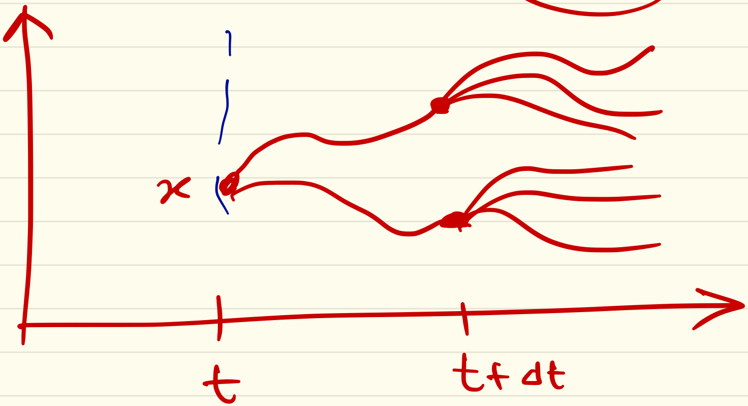
Principle of Optimality in Continuous Time:

For every $(t, \underline{x}) \in [0, T] \times \mathcal{X}$ and every $\Delta t \in (0, T-t]$, the value function $V(t, \underline{x})$ satisfies:

$$V(t, \underline{x}) = \inf_{\substack{u(s) \\ t \le s \le (t+\Delta t)}} \left\{ \left(\int_t^{t+\Delta t} L(s, x, u)\, ds \right) + V(t+\Delta t, \underline{x}(t+\Delta t)) \;\middle|\; x(t) = x,\ u(s) = u,\ t \le s \le t+\Delta t \right\} \quad (\star)$$

RHS

$$V(T, x) = \phi(T, x(T)).$$



Formal proof:

$V(t, x) \le \text{RHS}$

$V(t, x) \ge \text{RHS}.$

Earlier, the principle of optimality $\longrightarrow$ DP recursion (in discrete time)

Now, " " " " $\longrightarrow$ HJB PDE (in continuous time)

(Hamilton - Jacobi - Bellman)

Idea of deriving the HJB PDE:

To discretize time

$t \qquad t+\Delta t \qquad t+2\Delta t$

Then

$$x(t+\Delta t) = x + f(t, x, u(t))\Delta t + o(\Delta t)$$

where $x(t) = x$.

Then

$\boxed{\text{Assumption : } V \text{ is } \mathcal{C}^1(\mathcal{X})}$

$$V(t+\Delta t, x(t+\Delta t)) = V(t,x) + \frac{\partial V(t,x)}{\partial t}\Delta t + \left\langle \frac{\partial V(t,x)}{\partial x}, f(t,x,u \right. \Delta t + o(\Delta t) \quad \text{---} (**)$$

Also, $\displaystyle\int_t^{t+\Delta t} L(s, x, u)\, ds = L(t, x, u)\Delta t + o(\Delta t)$ --- ($)

Substitute (**) & ($) in (*) :

$$\underline{\underline{V(t, \underline{x})}} = \inf_{\substack{u(s) \in \mathcal{U} \\ t \le s \le t+\Delta t}} \Big\{ L(t, x, u)\Delta t + \boxed{V(t, x)} + \frac{\partial V}{\partial t}(t, x)\,\Delta t + \Big\langle \frac{\partial V}{\partial x}(t, x),\ f(t, x, u)\Delta t \Big\rangle + o(\Delta t) \Big\}$$

does not depend on "u"

$\Downarrow$

pull it out & cancel with the one in LHS

$$\Rightarrow 0 = \inf_{u \in \mathcal{U}} \Big\{ L(t, x, u)\Delta t + \frac{\partial V}{\partial t}\Delta t + \Big\langle \frac{\partial V}{\partial x},\ f(t, x, u)\Delta t \Big\rangle + o(\Delta t) \Big\}$$

$\therefore$ Dividing by Δt, and letting $\Delta t \to 0$,

we have $\dfrac{o(\Delta t)}{\Delta t} \to 0$.

$\Downarrow$

Pull out $\dfrac{\partial V}{\partial t}$ (since it does not depend on "u")

$\Downarrow$

$$-\frac{\partial V}{\partial t} = \inf_{\underline{u} \in \mathcal{U}} \left\{ L(t, \underline{x}, \underline{u}) + \left\langle \frac{\partial V}{\partial \underline{x}}, \underline{f}(t, \underline{x}, \underline{u}) \right\rangle \right\}$$

$\forall\, t \in [0, T]$

$\forall\, \underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$

Since $\Delta t \to 0$, this becomes a pointwise minimization

Terminal Condition

$$\Rightarrow \boxed{\frac{\partial V}{\partial t} + \underbrace{(\text{copy})}_{} = 0}, \qquad \boxed{V(T, \underline{x}(T)) = \phi(T, \underline{x}(T))}$$

$\mathcal{H}^*$ (minimized Hamiltonian)

Recall that $H := L + \lambda^T f$, i.e., $\dfrac{\partial V}{\partial \underline{x}}$ plays the role of costate $\lambda(t)$