Lecture #9

To solve the 2PBVP : consider ansatz (guess) : $\underline{\lambda}(t) = P(t)\underline{x}(t)$, $t \le T$

Recall: A, B, Q, R can be time-varying. (i.e.) $A(t), B(t), Q(t), R(t)$

Then,
$$\dot{\underline{\lambda}} = \dot{P}\underline{x} + P\dot{\underline{x}}$$
$$= \dot{P}\underline{x} + P(A\underline{x} - BR^{-1}B^TP\underline{x})$$

But LHS $= -Q\underline{x} - A^T\underline{\lambda}$
$$= -Q\underline{x} - A^TP\underline{x}$$

This gives :

$$-\dot{P}\underline{x} = (A^TP + PA - PBR^{-1}B^TP + Q)\underline{x}$$

But this must hold for all $\underline{x}_0$, hence for all $\underline{x}(t)$, where $0 \le t \le T$.

$$\therefore \boxed{-\dot{P} = A^TP + PA - PBR^{-1}B^TP + Q},$$

Riccati matrix ODE IVP

$\lambda(T) = P(T)x(T)$

$\Downarrow$

$Mx(T) = P(T)x(T)$

$\Uparrow$

$\boxed{P(T) = M \succeq 0}$

Strategy to solve finite-horizon LQR:

Back integrate Riccati matrix ODE Initial Value Problem (IVP)

$\downarrow$

use $P(T) = M$ to get $P(t)$, $0 \leq t \leq T$

$\downarrow$

get optimal control

$$u^*(x,t) = -\underbrace{R^{-1}B^T P(t)}\, \underline{x}(t)$$

$$= -\underbrace{K(t)}_{\text{Kalman gain}}\, \underline{x}(t)$$

optimal control is linear state feedback! [This is a result, not an assumption]

$$K(t) := R^{-1}B^T P(t)$$

$\downarrow$

Forward integrate state eq^n:

$$\dot{x} = A(t)\,x(t) + B(t)\,u^*(t), \quad \underbrace{\underline{x}(0) = x_0}_{\text{given}}$$

$\longrightarrow$ Get optimal state $\underline{x}^*(t)$

- Optimal costate trajectory: $\underline{\lambda}^*(t) = P(t)\,\underline{x}^*(t)$

- Closed-loop system:

$$\boxed{\dot{\underline{x}}(t) = \left(A - BK(t)\right) x(t)}$$

(closed-loop is LTV even if open loop is LTI)

- Sufficiency: $\nabla_u \circ \nabla_u J_{LQR} = R \succ 0$.

$\therefore$ Unique minimizing control.

Solving Quadratic Riccati Matrix ODE via linear Hamiltonian matrix ODE (a.k.a. Bernoulli substitution):

<u>Intuition:</u> $\underline{\lambda}(t) = P(t)\,\underline{x}(t)$

suggests that $P(t) = \underline{\lambda}(t)\left(\underline{x}(t)\right)^{-1}$

(nonsense unless $n=1$)

Now consider $\boxed{\text{linear}}$ Hamiltonian ODE in matrix (NOT vector) variables $X(t), \Lambda(t) \in \mathbb{R}^{n\times n}$

$$\begin{pmatrix} \dot{X} \\ \dot{\Lambda} \end{pmatrix} = \begin{bmatrix} A & -\underline{BR^{-1}B'} \\ -Q & -A^T \end{bmatrix} \begin{pmatrix} X \\ \Lambda \end{pmatrix}$$

with final condition $X(T) = I_n$, $\Lambda(T) = M \succeq 0$.

Theorem: $P(t) = \Lambda(t)\,(X(t))^{-1}$.

Proof: Let RHS $=:$ $\Psi(t) := \Lambda(t)\,(X(t))^{-1}$

We will show that $\Psi(t) \equiv P(t)$.

$$\dot\Psi = \underbrace{\dot\Lambda}_{\shortparallel} X^{-1} + \Lambda\left(\frac{d}{dt}X^{-1}\right)$$

$$= \left(-QX - A^T\Lambda\right)X^{-1}$$

$$\quad - \Lambda X^{-1}\left(AX - BR^{-1}B^T\Lambda\right)X^{-1}$$

$$= -Q - A^T\Psi - \underbrace{\Lambda X^{-1}}_{\Psi}A$$

$$\quad + \underbrace{\Lambda X^{-1}}_{\Psi}BR^{-1}B^T\underbrace{\Lambda X^{-1}}_{\Psi}$$

$$XX^{-1} = I$$

$$\Rightarrow \dot X X^{-1} + X\left(\frac{d}{dt}X^{-1}\right) = 0$$

$$\Rightarrow X\left(\frac{d}{dt}X^{-1}\right) = -\dot X X^{-1}$$

$$\Rightarrow \frac{d}{dt}\left(X^{-1}\right) = -X^{-1}\dot X X^{-1}$$

$$\therefore \quad \dot{\Psi} = -Q - A^T\Psi - \Psi A + \Psi B R^{-1} B^T \Psi$$

with $\Psi(T) = \underbrace{\Lambda(T)}_{= M} \big(\underbrace{X(T)}_{= I_n}\big)^{-1}$

$$= M \, (I_n)^{-1} = M$$

This is exactly our Riccati IVP.

$$\therefore \quad \boxed{\Psi(t) \equiv P(t)} \qquad \blacksquare$$

For LTI case:

$$\begin{pmatrix} X(t) \\ \Lambda(t) \end{pmatrix} = \underbrace{\exp\big(H(t-T)\big)}_{\Theta(t)} \begin{pmatrix} I_n \\ M \end{pmatrix}$$

$$= \begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix} \begin{pmatrix} I_n \\ M \end{pmatrix}$$

$$\therefore P(t) = \underbrace{\left(\Theta_{21}(t) + \Theta_{22}(t)\, M\right)}_{\Lambda(t)} \Big(\underbrace{\Theta_{11}(t) + \Theta_{12}(t) M}_{X(t)}\Big)^{-1}$$

LQR with cross weights :

If $L = \frac{1}{2} \underbrace{(\underline{x} \quad \underline{u})^T}_{1 \times (n+m)} \Pi \underbrace{\begin{pmatrix} \underline{x} \\ \underline{u} \end{pmatrix}}_{(n+m) \times 1}$

Popov matrix : $\Pi := \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \in \mathbb{S}_{+}^{(m+n)}$

S is called cross-weight matrix

The prev. derivation goes through.

Now you get $\underbrace{K(t) = R^{-1}B^T P(t) + S^T}_{\text{New Kalman gain}}$

Riccati ODE:

$$-\dot{P} = A^T P + PA - (PB+S)R^{-1}(PB+S)^T + Q.$$

previously, $S \equiv 0$.

New Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ -Q + SR^{-1}S^T & -A^T + SR^{-1}B^T \end{bmatrix}.$$

Finite Horizon LQR with terminal cost for tracking:

$$\dot{\underline{x}} = A\underline{x}(t) + B\underline{u}(t), \quad \underline{y}(t) = C\underline{x}(t), \quad 0 \le t \le T$$

Reference/desired trajectory to track:

$\underline{y}_d(t)$

Cost to minimize:

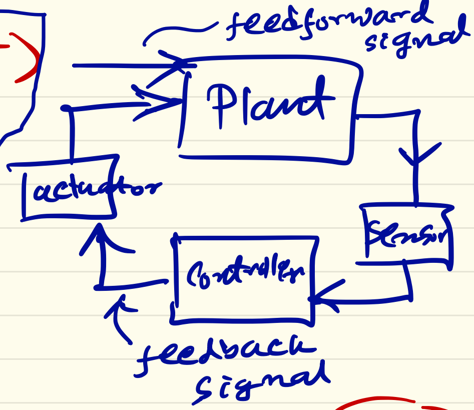
$$J = \frac{1}{2}\Big(\underbrace{\underline{y}(T)}_{C\underline{x}(T)} - \underline{y}_d(T)\Big)' M \Big(\underbrace{\underline{y}(T)}_{C\underline{x}(T)} - \underline{y}_d(T)\Big) +$$

$$\int_0^T \Big\{ \Big(\overbrace{\underline{y}(t)}^{C\underline{x}(t)} - \underline{y}_d(t)\Big)^T Q \Big(\overbrace{\underline{y}(t)}^{C\underline{x}(t)} - \underline{y}_d(t)\Big) + u^T R u \Big\} dt$$

Exercise: show that:

$$u^*(x, t) = u^*_{\text{feedback}}(\underline{x}(t)) + u^*_{\text{feedforward}}(t)$$

$$\boxed{u^*_{\text{feedback}}(\underline{x}(t)) = -\underbrace{K(t)}_{\text{Kalman gain}}\, \underline{x}(t)}$$



Where $K(t) := R^{-1} B^T P(t)$

Riccati ODE:

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + C^T Q C$$

terminal condition: $P(T) = C^T M C$

$$\boxed{u^*_{\text{feedforward}}(t) = R^{-1} B^T \underline{v}(t)}$$

where $\underline{v}(t)$ solves feedforward ODE:

$$-\dot{v}(t) = (A - BK)' v(t) + C^T Q \underline{y}_d(t)$$

terminal cond^{n}: $v(T) = C^T M \underline{y}_d(T)$

- Optimal Cost for finite horizon LQR:

We use $*$ for optimal
$'$ " matrix transpose
T " final time

$$J^* = \frac{1}{2}\left\{\underline{x}^{*\prime}(T)\, M\, \underline{x}^*(T) + \int_0^T \left(\underline{x}^{*\prime} Q\, \underline{x}^* + \underline{u}^{*\prime} R\, \underline{u}^*\right) dt\right\}$$

$\{u^* = -K(t)\, x^*$

$$= \frac{1}{2}\left\{\underline{x}^{*\prime}(T)\, M\, \underline{x}^*(T) + \int_0^T \left[\underline{x}^{*\prime}\left(Q + K'(t)\, R\, K(t)\right)\underline{x}^*\right] dt\right\}$$

$$= \frac{1}{2}\left\{\underline{x}^{*\prime}(T)\, M\, \underline{x}^*(T) + \int_0^T \left[\underline{x}^{*\prime}\left(Q + PBR^{-1}P\right)\underline{x}^*\right] dt\right\}$$

Now consider:

$$\frac{d}{dt}\left(\underline{x}^{*\prime}\, P(t)\, \underline{x}^*\right) = \left(\underline{\dot{x}}^*\right)' P(t)\, \underline{x}^* + \underline{x}^{*\prime}\, \dot{P}\, \underline{x}^* + \underline{x}^{*\prime}\, P\, \underline{\dot{x}}^*$$

$$= (\underline{x}^*)'(A - BK)'P\,\underline{x}^* \;+$$

$$(\underline{x}^*)'\left(-A^TP - PA + PBR^{-1}B^TP - Q\right)\underline{x}^* \;+$$

$$(\underline{x}^*)'\,P\,(A - BK)\,\underline{x}^*$$

$$= (\underline{x}^*)'\Big\{(A - BK)'P - A'P - PA + PBR^{-1}B'P - Q + P(A - BK)\Big\}\underline{x}^*$$

sub. for $K(t)$ (Kalman gain)

$$= (\underline{x}^*)'\Big\{\cancel{A'P} - \underbrace{K'B'}\,P - \cancel{A'P} - \cancel{PA} + PBR^{-1}B'P - Q + \cancel{PA} - PBK\Big\}\,\underline{x}^*$$

$$= (\underline{x}^*)'\Big\{\cancel{-PBR^{-1}B^TP} + \cancel{PBR^{-1}B'P} - Q - PBR^{-1}B^TP\Big\}\,\underline{x}^*$$

$$= (\underline{x}^*)'\Big\{-Q - PBR^{-1}B^TP\Big\}\,\underline{x}^*$$

$$\therefore J^* = \frac{1}{2}\left\{ \underline{x}^{*\prime}(T) M \underline{x}^*(T) - \int_0^T \frac{d}{dt}\left(\underline{x}^{*\prime} P(t) \underline{x}^*\right) dt \right\}$$

$$= \frac{1}{2}\left\{ \underline{x}^{*\prime}(T) M \underline{x}^*(T) - \int_0^T d\left(\underline{x}^{*\prime} P(t) \underline{x}^*\right) \right\}$$

$$= \frac{1}{2}\left\{ \cancel{\underline{x}^{*\prime}(T) M \underline{x}^*(T)} - \cancel{\underline{x}^{*\prime}(T) \underbrace{P(T)}_{M} \underline{x}^*(T)} \right.$$

$$\left. + \left(\underline{x}^*(0)\right)' P(0) \underline{x}^*(0) \right\}$$

$$= \underbrace{\boxed{\frac{1}{2}\left(\left(\underline{x}(0)\right)' P(0) \underline{x}(0)\right)}}$$

Since $\underline{x}(0)$ is given, we can drop the superscript $*$ from the initial condition

Intuitive: The farther we start, the larger becomes the optimal cost.

Properties of Riccati ODE:

(I) Under our hypotheses ($M \succeq 0, Q \succeq 0, R \succ 0$) existence & uniqueness for $P(t)$ is guaranteed over $0 \leq t \leq T$

(II) For any $t_0 \leq t \leq T$, and any $M \succeq 0$, we have $P(t) \succeq 0$.

Proof for (II): By transposing the Riccati ODE, notice that if $P(t)$ is a sol^{n}, then so is $P'(t)$. But $\exists$ unique sol^{n}.

$\therefore P(t) = P'(t) \Rightarrow P(t)$ must be a symm. matrix

On the other hand:

$0 \leq J^* = \frac{1}{2}(\underline{x}(0))' P(0) \underline{x}(0) \quad \forall \; \underline{x}(0)$

$\therefore \; P(0) \succeq 0$

But t_0 was arbitray.

$$\therefore \; P(t) \succcurlyeq 0 \quad \forall \; t \leq T. \quad \blacksquare$$

Next, we look into the discrete-time OCP & its necessary conditions for optimality

We will illustrate those on discrete-time LQR

Discrete Time OCP and Necessary Conditions:

- Let discrete time index $k = 0, 1, 2, \ldots, N$
- If $N < \infty$, we say it is a discrete-time finite horizon problem
- OCP: $\min\limits_{\{u_k\}_{k=0}^{N-1}} J$, where $J := \overbrace{\phi(\underline{x}_N, N)}^{\text{terminal cost}} + \sum\limits_{k=0}^{N-1} \overbrace{L}^{\text{Lagrangian}} \underbrace{(\underline{x}_k, \underline{u}_k, k)}_{\underset{L_k}{\shortparallel}}$

s.t.

$$\underline{x}_{k+1} = \underbrace{\underline{f}(\underline{x}_k, \underline{u}_k, k)}_{=:\, \underline{f}_k} \Bigg\} \text{ Discrete-time controlled dynamics}$$

- Hamiltonian $H_k := L_k + \underline{\lambda}_{k+1}^T \underline{f}_k$
- $\boxed{\text{State eq}^{n}}$ $\underline{x}_{k+1} = \dfrac{\partial H_k}{\partial \underline{\lambda}_{k+1}} = \underline{f}(\underline{x}_k, \underline{u}_k, k)$
- $\boxed{\text{Costate eq}^{n}.}$ $\underline{\lambda}_k = \dfrac{\partial H_k}{\partial \underline{x}_k}$

- $\boxed{\text{PMP}}$ $\quad \underline{0} = \dfrac{\partial H_k}{\partial \underline{u}_k}$

- $\boxed{\text{Boundary conditions (Plays the role of transversality)}}$

- $\left(\dfrac{\partial L_0}{\partial \underline{x}_0} + \left(\dfrac{\partial \underline{f}_0}{\partial \underline{x}_0}\right)^T \underline{\lambda}_1 \right)^T d\underline{x}_0 = 0$

- $\left(\dfrac{\partial \phi}{\partial \underline{x}_N} - \underline{\lambda}_N \right)^T d\underline{x}_N = 0$

We will henceforth assume that $\underline{x}_0$ is fixed (true for all practical problems). $\therefore d\underline{x}_0 = 0$, i.e., the first equality always holds.

Compare these necessary conditions with the corresponding continuous-time conditions.

Discrete Time Finite Horizon LQR with Terminal Cost

$M \succeq 0$, $Q \succeq 0$, $R \succ 0$

$$\min_{\{\underline{u}_k\}_{k=0}^{N-1}} \frac{1}{2}\left\{\underline{x}_N^T M \underline{x}_N + \sum_{k=0}^{N-1}\left(\underline{x}_k^T Q \underline{x}_k + \underline{u}_k^T R \underline{u}_k\right)\right\}$$

$$\text{s.t.} \quad \underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k, \quad \underline{x}(0) = \underline{x}_0 \text{ given}$$

- Again (A, B, Q, R) may be time-varying, i.e., may have subscript k.
- To solve LQR means to find the control sequence $\{\underline{u}_0^*, \underline{u}_1^*, \ldots, \underline{u}_{N-1}^*\}$

In this case, Hamiltonian

$$H_k = \frac{1}{2}\underline{x}_k^T Q \underline{x}_k + \frac{1}{2}\underline{u}_k^T R \underline{u}_k + \underline{\lambda}_{k+1}^T (A\underline{x}_k + B\underline{u}_k)$$

- Costate eq^{n}: $\underline{\lambda}_k = \frac{\partial H_k}{\partial \underline{x}_k} = Q\,\underline{x}_k + A^T \underline{\lambda}_{k+1}$

- PMP: $\underline{0} = \frac{\partial H_k}{\partial \underline{u}_k} = R\,\underline{u}_k + B^T \underline{\lambda}_{k+1}$

$$\Rightarrow \boxed{\underline{u}_k = -R^{-1}B^T \underline{\lambda}_{k+1}}$$

- Boundary cond^{n}: $d\underline{x}_N \neq 0$

$$\Rightarrow \frac{\partial \phi}{\partial \underline{x}_N} - \underline{\lambda}_N = 0$$

$$\Rightarrow M\underline{x}_N - \underline{\lambda}_N = 0 \Rightarrow \boxed{\underline{\lambda}_N = M\underline{x}_N}$$

Therefore, in this case, we have 2PBVP:

$$\begin{pmatrix} \underline{x}_{k+1} \\ \underline{\lambda}_k \end{pmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ Q & A^T \end{bmatrix}}_{\text{Hamiltonian matrix}} \begin{pmatrix} \underline{x}_k \\ \underline{\lambda}_{k+1} \end{pmatrix}$$

$$\boxed{\begin{aligned} \underline{x}(0) &= \underline{x}_0 \\ \underline{\lambda}(N) &= M\underline{x}_N \end{aligned}}$$

To solve the 2PBVP, we consider the ansatz: $\underline{\lambda}_k = P_k \underline{x}_k$ for all k.

Now, $\underline{\lambda}_N = P_N \underline{x}_N$

$$\Rightarrow \underbrace{M x_N}_{\text{(from boundary condition)}} = P_N \underline{x}_N \Rightarrow \boxed{P_N = M \succeq 0}$$

Now, $\underline{\lambda}_{k+1} = P_{k+1} \underline{x}_{k+1}$

$$= P_{k+1} \overbrace{(A\underline{x}_k + B\underline{u}_k)}^{\text{state eq}^n}$$

from PMP

$$= P_{k+1}(A\underline{x}_k - BR^{-1}B^T \underline{\lambda}_{k+1})$$

$$\Rightarrow \boxed{\underline{\lambda}_{k+1} = (I + P_{k+1} B R^{-1} B^T)^{-1} P_{k+1} A \underline{x}_k}$$

On the other hand:

substitute for λ_{k+1}

$$\boxed{\text{Costate eq}^n} \quad \underbrace{\underline{\lambda}_k}_{\text{III}} = Q\underline{x}_k + A^T \underline{\lambda}_{k+1}$$

$$\Rightarrow P_k \underline{x}_k = Q\underline{x}_k + A^T (I + P_{k+1} B R^{-1} B^T)^{-1} P_{k+1} A \underline{x}_k$$

Since the last line of the prev. page must hold for all $\underline{x}_0$, and hence for all $\underline{x}_k$, therefore, we must have:

$$P_k = Q + A^T (I + P_{k+1} B R^{-1} B^T)^{-1} P_{k+1} A$$

$$= Q + A^T P_{k+1}^{1/2} (I + P_{k+1}^{1/2} B R^{-1} B^T P_{k+1}^{1/2})^{-1} P_{k+1}^{1/2} A$$

slightly more symmetric form

Riccati Matrix Recursion to be run backward in time from $k = N-1$ to $k=0$, with terminal condition: $P_N = M \succeq 0$

Clearly, $P_k \succeq 0$ for all k

$\therefore$ This is a nonlinear recursion on the pos. semi def. cone

Get $P_{k+1} \longrightarrow$ Get $\underline{\lambda}_{k+1} = P_{k+1} \underline{x}_{k+1} \longrightarrow$ Get $\underline{u}_k^* = -R^{-1} B^T \underline{\lambda}_{k+1}$

$\downarrow$

Get $\underline{x}^*$ by running state recursion

Notice that the optimal control:

$$\underline{u}_k^* = -R^{-1}B^T \underline{\lambda}_{k+1}$$
$$= -R^{-1}B^T P_{k+1}\underline{x}_{k+1}$$
$$= -R^{-1}B^T P_{k+1}(A\underline{x}_k + Bu_k^*)$$

$$\Rightarrow \left(I + R^{-1}B^T P_{k+1} B\right)\underline{u}_k^* = -R^{-1}B^T P_{k+1} A\,\underline{x}_k$$

$$\Rightarrow \quad \underline{u}_k^* = -\left(I + R^{-1}B^T P_{k+1}B\right)^{-1} R^{-1}B^T P_{k+1} A\,\underline{x}_k$$
$$= -\left(R^{-1}R + R^{-1}B^T P_{k+1} B\right)^{-1} R^{-1} B^T P_{k+1} A\,\underline{x}_k$$
$$= -\left(R + B^T P_{k+1} B\right)^{-1} \cancel{R}\,\cancel{R^{-1}}\, B^T P_{k+1} A\,\underline{x}_k$$
$$= -\underbrace{K_k}_{\text{Kalman gain}}\,\underline{x}_k$$

where

$$K_k := \left(R + B^T P_{k+1} B\right)^{-1} B^T P_{k+1} A$$