Lecture #8

Solving Min. Energy State Transfer :

$$H = L + \underline{\lambda}^T \underline{f}$$

$$= \frac{1}{2} \underline{u}^T \underline{u} + (\lambda(t))^T (A(t)\underline{x} + B(t)\underline{u})$$

$$\underline{\dot{x}} = \frac{\partial H}{\partial \underline{\lambda}} = A(t)\underline{x} + B(t)\underline{u},$$

$$\boxed{\underline{\dot{\lambda}} = -\frac{\partial H}{\partial \underline{x}} = -(A(t))^T \underline{\lambda}} \Leftrightarrow \boxed{\lambda(t) = -(\Phi(1,t))^T \lambda(1)}$$

e.g. If LTI, $A(t) \equiv A$.

$\Updownarrow$

$\lambda(t) = \exp(A^T(1-t))$

PMP

$$\underset{m\times 1}{\underline{0}} = \frac{\partial H}{\partial \underline{u}} = \underline{u} + (B(t))^T \underline{\lambda}(t)$$

$$\Rightarrow \boxed{u(t) = -(B(t))^T \underline{\lambda}(t)}$$

$$= +(B(t))^T (\Phi(1,t))^T \underline{\lambda}(1)$$

Substituting $\underset{\sim}{u}(t)$ from prev. page in state eqn: (optimal control)

and integrating: (prime $\equiv$ matrix transpose)

$$\dot{\underline{x}} = A(t)\underline{x}(t) + B(t)(B(t))'(\Phi(1,t))'\underline{\lambda}(1)$$

$$\Rightarrow \underline{x}(t) = \Phi(t,\cancel{t_0}\,0)\,x(\cancel{t_0}\,0) + \int_{\cancel{t_0}\,0}^{t} \Phi(t,\tau)B(\tau)B'(\tau)\,\Phi'(1,\tau)\underline{\lambda}(1)\,d\tau$$

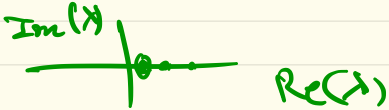
To get $\underline{\lambda}(1)$, evaluate the above @ $t=1$:

$$\underline{x}_1 = \Phi(1,0)\underline{x}_0 + \int_0^1 \Phi(1,\tau)B(\tau)B'(\tau)\Phi'(1,\tau)\underline{\lambda}(1)\,d\tau$$

$$\Leftrightarrow \underline{x}_1 - \Phi(1,0)\underline{x}_0 = \underbrace{M(0,1)}_{\equiv M_{01}}\underline{\lambda}(1)$$

$$\Leftrightarrow \underline{\lambda}(1) = M_{01}^{-1}\left[\underline{x}_1 - \Phi(1,0)\underline{x}_0\right]$$

$$\therefore \underline{u}^*(t) = B'(t)\Phi'(1,t)\underbrace{M_{01}^{-1}}_{\text{could be large.}}\left[\underline{x}_1 - \Phi(1,0)\underline{x}_0\right]$$



Optimal state trajectory :

$$\underline{x}^*(t) = \Phi(t,0)\underline{x}_0 + \int_0^t \Phi(t,\tau) B(\tau) B'(\tau) \Phi'(1,\tau) M_{01}^{-1} [\underline{x}_1 - \Phi(1,0)\underline{x}_0] \, d\tau$$

$$= \Phi(t,0)\underline{x}_0 + \int_0^t \Phi(t,\tau) B(\tau) B'(\tau) \Phi'(t,\tau) (\Phi'(t,\tau))^{-1} \Phi'(1,\tau) M_{01}^{-1} [\underline{x}_1 - \Phi(1,0)\underline{x}_0] \, d\tau$$

This product

$$= (\Phi'(t,\tau))^{-1} \Phi'(1,\tau)$$

used $\Phi^{-1}(t,\tau) = \Phi(\tau,t)$

$$= \left(\Phi(1,\tau) \underbrace{\Phi^{-1}(t,\tau)}_{=\Phi(\tau,t)} \right)'$$

$$= \left(\Phi(1,\tau) \Phi(\tau,t) \right)'$$

used semi-group property

$$= (\Phi(1,t))' = \Phi'(1,t)$$

$$\Rightarrow \underline{x}^*(t) = \underbrace{\Phi(t,0)\,\underline{x}_0}_{\text{term 1}} + \underbrace{\int_0^t \Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)\,\Phi'(1,t)M_{10}^{-1}x_1\,d\tau}_{\text{term 2}}$$

just proved

$$\underbrace{-\int_0^t \Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)\Phi'(1,t)M_{01}^{-1}\Phi(1,0)\,\underline{x}_0\,d\tau}_{\text{term 3}}$$

Combine terms 1 & 3 :

$$= \left\{\Phi(t,0) - M(0,t)\Phi'(1,t)M_{01}^{-1}\Phi(1,0)\right\}\underline{x}_0$$

$$+ M(t,0)\Phi'(1,t)M_{01}^{-1}\underline{x}_1$$

$$\boxed{\therefore\ \underline{x}^*(t) = \Phi(t,1)M(t,1)M_{01}^{-1}\Phi_{10}\,\underline{x}_0 + M(t,0)\Phi'(1,t)M_{01}^{-1}\underline{x}_1}$$

next pg.

How did we prove that the blue underlined expressions are equal?

Notice that the blue underlined expression in the penultimate line in the prev. page equals

$$\left\{ \Phi(t,0) \underbrace{\Phi^{-1}(1,0)}_{=\Phi(0,1)} M_{01} - M(0,t)\Phi'(1,t) \right\} M_{01}^{-1} \Phi(1,0)\underline{x}_0$$

$$= \left\{ \underbrace{\Phi(t,0)\Phi(0,1)}\, M_{01} - \underbrace{\Phi(t,1)(\Phi(t,1))^{-1}}_{\text{equals Identity}} M(0,t)\Phi'(1,t) \right\} M_{10}^{-1}\Phi(1,0)\underline{x}_0$$

$$= \left\{ \Phi(t,1) M_{01} - \Phi(t,1)\Phi(1,t) M(0,t)\Phi'(1,t) \right\} M_{10}^{-1}\Phi(1,0)\underline{x}_0$$

$$= \Phi(t,1) \left\{ \underbrace{M_{01} - \Phi(1,t) M(0,t)\Phi'(1,t)} \right\} M_{10}^{-1}\Phi(1,0)\underline{x}_0$$

Let us now concentrate on this term - - - - - (*)

claim: $M_{01} = M(t,1) + \Phi(1,t) M(0,t)\Phi'(1,t)$ - - - - (**)

next pg.

Proof of claim:

$$\text{LHS} = M_{01} := \int_0^1 \Phi(1,\tau)\, B(\tau)\, B'(\tau)\, \Phi'(1,\tau)\, d\tau$$

$$= \int_0^t \cdots\cdots\, d\tau + \underbrace{\int_t^1 \cdots\cdots\, d\tau}_{= M(t,1)}$$

$$= \left\{ \int_0^t \boxed{\Phi(1,\tau)}\, B(\tau)\, B'(\tau)\, \boxed{\Phi'(1,\tau)}\, d\tau \right\} + M(t,1)$$

$$= \left\{ \int_0^t \boxed{\Phi(1,t)\,\Phi(t,\tau)}\, B(\tau)\, B'(\tau)\, \boxed{\Phi'(t,\tau)\ \Phi'(1,t)}\, d\tau \right\} + M(t,1)$$

apply semigroup property for STM

peel off $\Phi(1,t)$

$$= \Phi(1,t) \underbrace{\left\{ \int_0^t \Phi(t,\tau)\, B(\tau)\, B'(\tau)\, \Phi'(t,\tau)\, d\tau \right\}}_{= M(0,t)} \Phi'(1,t) + M(t,1)$$

$\blacksquare$

Substituting (**) in (*), we get the desired expression for optimal state trajectory $\underline{x}^*(t)$: given 2 pages back (in the green box).

Our key step in the above derivation was the claim (**). We can generalize that claim & its proof as the following powerful Lemma.

Lemma : Consider $t_0 < t < t_1$.

Then,

$$\boxed{M(t_0, t_1) = M(t, t_1) + \Phi(t_1, t) M(t_0, t) \Phi'(t_1, t)}$$

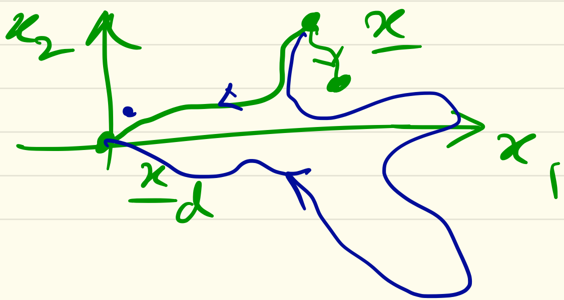
Remark : Our claim proved in the prev. page above Lemma, is a special instance of the above Lemma, (i.e.) $t_0 \equiv 0$, $t_1 \equiv 1$.

To get $u^* = fn(x^*)$ [optimal feedback form],
we need to express
$\underline{x}_0$ as f^n of $\underbrace{\underline{x}^*}_{\text{from sensor}}$ and $\underbrace{M(t,0)}_{\text{from Lyapunov ODE sol}^n}$.
(or $\underline{x}_1$)

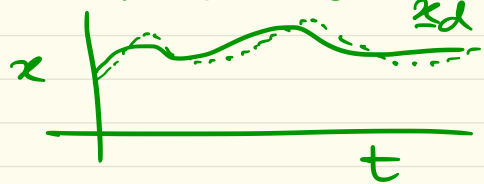
[End of Example]

LQR
(Linear Quadratic Regulator)

Regulation Problem
(Desired Point)



Tracking Problem
(Desired trajectory)



Linear dynamics: $\dot{\underline{x}}(t) = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t)$,

$\underline{x} \in \mathbb{R}^n$

$\underline{u} \in \mathbb{R}^m$

Quadratic cost

$$\underset{u(\cdot)}{\text{minimize}}\ \frac{1}{2}\left\{ \underbrace{(x(T))' M\, x(T)}_{\text{terminal cost}} + \int_0^T \left\{ \underbrace{x' Q(t)\, x}_{\text{state effort}} + \underbrace{u' R(t) u}_{\text{control effort}} \right\} dt \right\}$$

(i.e.) $\phi(x(T), T) = \frac{1}{2}(x(T))' M\, x(T)$ } Terminal cost

$L = \frac{1}{2}\{x' Q(t)\, x + u' R(t)\, u\}$ Lagrangian

In this problem, final time (T) is fixed (for now)

No terminal constraint

The matrices: (weight matrices) $M \succcurlyeq 0$

$Q(t) \succcurlyeq 0$, $R(t) \succ 0$ for $t \in [0,T]$

Hamiltonian : (Let's drop the "t" for notational ease)

$$H = \frac{1}{2}\left(\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u}\right) + \underline{\lambda}^T \left(A\underline{x} + B\underline{u}\right)$$

$$\dot{\underline{\lambda}} = -\nabla_{\underline{x}} H = Q\underline{x} + A^T \underline{\lambda}$$

PMP

$$\underline{0}_{m\times 1} = \nabla_{\underline{u}} H = Ru + B^T \underline{\lambda}$$

$$\Rightarrow \underline{u}(t) = -R^{-1} B^T \underline{\lambda}(t)$$

Transversality : $dT = 0$

$d\underline{x}(T) \neq 0$

$$\Rightarrow \boxed{\underline{\lambda}(T) = M x(T)}$$

$$\underline{z} := \underline{x} - \underline{x}_d$$

state running cost

$$(\underline{x} - \underline{x}_d)^T Q(t) (\underline{x} - \underline{x}_d) = \underline{z}^T Q(t) \underline{z}$$

Terminal Cost

$$(\underline{x} - \underline{x}_d)^T M (\underline{x} - \underline{x}_d) = \underline{z}^T M \underline{z}$$

So the same LQR problem formulation can be used to regulate the state to an arbitrary desired pt. $\underline{x}_d \in \mathbb{R}^n$

$\therefore \quad \dot{\underline{x}} = Ax + Bu$

$\quad = Ax - BR^{-1}B^T\lambda$

$\dot{\underline{\lambda}} = Qx + A^T\lambda$

$x(0) = \underline{x}_0$ given

$\underline{\lambda}(T) = M\underline{x}(T)$

$$\underbrace{\begin{pmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{pmatrix}}_{2n\times 1} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{2n\times 2n} \underbrace{\begin{pmatrix} \underline{x} \\ \underline{\lambda} \end{pmatrix}}_{2n\times 1},$$

2PBVP (Two point Boundary value Problem).