Lecture #7

Intercept problem (contd.)

Problem:

$$\underbrace{\underset{\gamma(t)}{\text{minimize}} \int_0^T 1 \, . \, dt = T}_{\text{minimum time intercept}}$$

s.t.

$$\left.\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= a\cos\gamma \\ \dot{v} &= a\sin\gamma \end{aligned}\right| \quad \begin{aligned} &a = \text{constant (given)} > 0 \\ &\text{I.C. } x(0) = y(0) = u(0) = v(0) \\ &\qquad\qquad\qquad = 0. \end{aligned}$$

Apply 1st order conditions:

$$H = L + \underline{\lambda}^T f$$

$$= 1 + \lambda_1(t) u + \lambda_2(t) v + \lambda_3(t) a\cos\gamma + \lambda_4(t) a\sin\gamma$$

Condⁿ ① : (state dynamics) $\dot{\underline{x}} = f(\underline{x}, \gamma)$ already known.

Condⁿ ② (costate dynamics) :

$$\left.\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x} = 0 \iff \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial y} = 0 \iff \\
\dot{\lambda}_3 &= -\frac{\partial H}{\partial u} = -\lambda_1 \iff \\
\dot{\lambda}_4 &= -\frac{\partial H}{\partial v} = -\lambda_2 \iff
\end{aligned}\right\} \left\{\begin{aligned}
&\lambda_1 = \text{const.} = \nu_1 \\
&\lambda_2 = \text{const.} = \nu_2 \\
&\lambda_3(t) = (T-t)\nu_1 \\
&\lambda_4(t) = (T-t)\nu_2
\end{aligned}\right\}$$

$\forall\, t \in [0, T]$

Condⁿ ③ : (PMP)

$$\boxed{0 = \frac{\partial H}{\partial \gamma} = -\lambda_3\, a \sin\gamma^* + \lambda_4\, a \cos\gamma^*}$$

Condⁿ ④ (Transversality)

Final time free ($\iff T$ free), Final state is also free ($\iff \underline{x}(T)$ free) we have terminal constraint $\underline{\Psi}$:

$$\left.\begin{aligned} \therefore\ & dT \neq 0 \\ & d\underline{x}(T) \neq 0 \end{aligned}\right\} \text{from transversality :}$$

$$\underset{2\times 1}{\underline{\Psi}}\left(\underline{x}(T), T\right) = \begin{pmatrix} x(T) - (x_0 + VT) \\ y(T) - h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Transversality :

$$\left(\left.\frac{\partial \underline{\Psi}_{2\times 1}}{\partial \underline{x}_{4\times 1}}\right|_{t=T}\right)^{T} \underset{2\times 1}{\underline{\nu}} - \underbrace{\left.\underline{\lambda}\right|_{t=T}}_{4\times 1} = \underset{4\times 1}{\underline{0}} \quad \text{-----}(1)$$

$$\left(\left.\frac{\partial \underline{\Psi}_{2\times 1}}{\partial t}\right|_{t=T}\right)^{T} \underset{2\times 1}{\underline{\nu}} + H(T) = 0 \quad \text{----}(2)$$

$\therefore$ (1) becomes:

$$\underline{\lambda}(T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_{2\times 1}$$

$$\Leftrightarrow \boxed{\begin{aligned} \lambda_1(T) &= \nu_1 \\ \lambda_2(T) &= \nu_2 \\ \lambda_3(T) &= 0 \\ \lambda_4(T) &= 0 \end{aligned}}$$

On the other hand, (2) becomes:

$$H(T) = - \left\langle \frac{\partial \psi}{\partial t}, \underline{\nu} \right\rangle \Big|_{t=T}$$

$$= - \begin{bmatrix} -V & 0 \end{bmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = +V\nu_1$$

$$\Rightarrow 1 + \cancelto{\nu_1}{\lambda_1(T)}\, u(T) + \cancelto{\nu_2}{\lambda_2(T)}\, v(T) + \cancelto{0}{\lambda_3(T)}\, a\cos\gamma(T) + \cancelto{0}{\lambda_4(T)}\, a\sin\gamma(T)$$

$$= +V\nu_1$$

$$\Rightarrow \boxed{1 + \nu_1 u(T) + \nu_2 v(T) = V \nu_1}$$

From PMP:

$$\tan(\gamma^*(t)) = \frac{\lambda_4(t)}{\lambda_3(t)} = \frac{\nu_2 \cancel{(T-t)}}{\nu_1 \cancel{(T-t)}}$$

$$\Rightarrow \tan \gamma^*(t) = \frac{\nu_2}{\nu_1}$$

$\Rightarrow$ Optimal thrust angle is constant.

What remains is to compute ν_1 & ν_2:

Integrate the state eq^n:

$$\therefore \left.\begin{aligned} u^*(t) &= at\cos(\gamma^*) \\ v^*(t) &= at\sin(\gamma^*) \\ x^*(t) &= \frac{at^2}{2}\cos(\gamma^*) \\ y^*(t) &= \frac{at^2}{2}\sin(\gamma^*) \end{aligned}\right\} \Rightarrow \tan\gamma^* = \frac{y^*(t)}{x^*(t)} = \frac{y^*(T)}{x^*(T)}$$

$$\Rightarrow \boxed{\tan \gamma^* = \frac{y^*(T)}{x^*(T)} = \frac{h}{x_0 + VT^*}}$$

(still NOT done, need to find T^*)

(optimal terminal time)

we are using:

$$x_0 + VT^* = x^*(T) = \frac{aT^{*2}}{2} \cos(\gamma^*)$$

$$\Rightarrow \cos(\gamma^*) = \frac{2(x_0 + VT^*)}{aT^{*2}}$$

and

$$h = y^*(T) = \frac{aT^{*2}}{2} \sin(\gamma^*)$$

$$\Rightarrow \sin(\gamma^*) = \frac{2h}{aT^{*2}}$$

$$\boxed{\cos^2(\gamma^*) + \sin^2(\gamma^*) = 1 \text{ gives}}$$

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$$\left(-\frac{a^2}{4}\right)(T^*)^4 + V^2 (T^*)^2 + (2Vx_0)\, T^* + (x_0^2+h^2) = 0$$

quartic eqn. in T^*

has unique positive root $T^* > 0$

Descartes' rule of sign:

Coefficient sign sequence:

exactly one change in sign

$$\underbrace{- \quad , \quad +}_{\curvearrowright} \quad , \quad + \quad , \quad +$$

$\therefore$ 1 positive root (unique $T^* > 0$)

Example: Minimum Energy State Transfer
(Minimum norm)

Final time $T = 1$ = fixed

$$\min_{\underline{u}(\cdot)} J(\underline{u}) = \frac{1}{2}\int_0^T \underline{u}^T \underline{u} \, dt$$

Final state $\underline{x}(T) = \underline{x}_1$ (given)

(fixed)

$$= \int_0^T \frac{1}{2} \|\underline{u}\|_2^2 \, dt$$

Transversality: $0 + 0 = 0$ (no new information)

s.t.

$$\underline{\dot{x}} = \overset{n \times n}{A(t)} \, \underline{x} + \overset{n \times m}{B(t)} \, \overset{n \times 1}{\underline{u}} \, , \quad \underline{x} \in \mathbb{R}^n, \quad \underline{u} \in \mathbb{R}^m$$

I.C. $\underline{x}(0) = x_0$ given

$\underline{x}(1) = \underline{x}_1$ given, $T = 1$ fixed.

The dynamics

$$\dot{\underline{x}}(t) = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t)$$ } Linear time-varying system

(LTV system)

Spl. case: $\underline{\text{LTI}}$ system: $(A(t), B(t)) = (A, B)$

(Time Invariant)

Background on State Transition Matrix & Controllability Gramian

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State Transition Matrix (STM)

$\Phi(t,s)$, e.g. $\dot{\underline{x}} = A(t)\,\underline{x}(t)$, $\underline{x}(0) = \underline{x}_0$.

$0 \le s < t < 1$

$$\underbrace{\underline{x}(t)}_{n\times 1} = \underbrace{\Phi(t,0)}_{n\times n}\,\underbrace{\underline{x}_0}_{n\times 1}$$

(sol$^{\underline{n}}$ of homogeneous linear ODE)

For LTI:

$$\Phi(t,s) = \exp(A(t-s))$$

$$= I + A(t-s) + \frac{(t-s)^2}{2!}A^2 + \cdots$$

Recall that for $\boxed{m \neq n}$

$$\underline{\dot{x}} = A(t)\,\underline{x}(t) + \overset{n\times m}{B(t)}\,\overset{m\times 1}{\underline{u}(t)}, \quad \underline{x}(t_0) = \underline{x}_0^{\text{(given)}}$$

$$\Rightarrow \underbrace{\underline{x}(t)}_{n\times 1} = \underbrace{\Phi(t,t_0)}_{n\times n}\,\underbrace{\underline{x}_0}_{n\times 1} + \int_{t_0}^{t} \underbrace{\underbrace{\Phi(t,\tau)}_{n\times n}\,\underbrace{B(\tau)}_{n\times m}\,\underbrace{u(\tau)}_{m\times 1}\,d\tau}_{n\times 1}$$

(solution of non-homogeneous linear ODE.

Properties of STM :

- $\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0)$
- $\Phi(\tau,\tau) = I$
- $\Phi^{-1}(t,\tau) = \Phi(\tau,t)$
- $\Phi(t_2,t_1)\,\Phi(t_1,t_0) = \Phi(t_2,t_0)$

Controllability Gramian

an n×n matrix

$m \neq n$

Here prime denotes "transpose".

$$M(s,t) := \int_s^t \Phi(t,\tau)\, B(\tau)\, B'(\tau)\, \Phi'(t,\tau)\, d\tau,$$

(Definition)

$\succeq 0$ (always)

$0 \leq s < t \leq 1$

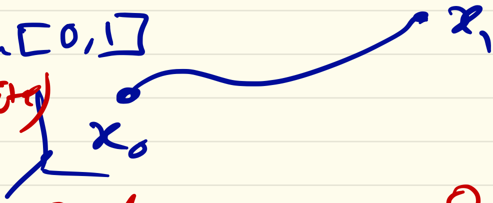
Following statements are equivalent:

(I) The system $(A(t), B(t))$ is controllable in time interval $[0,1]$

$\Updownarrow$

(II) M is non-singular in $[0,1]$

$\Leftrightarrow M \succ 0$ (positive definite)



M(s,t) solves linear Matrix ODE (Lyapunov ODE):

$$\boxed{\dot{M} = A(t)M + M(A(t))' + B(t)B'(t),}$$

Proof that M(s,t) satisfies the Lyapunov ODE :

$$\frac{d}{dt} M(s,t) = \frac{d}{dt} \int_s^t \Phi(t,\tau) B(\tau) B'(\tau) \Phi'(t,\tau) \, d\tau$$

$$= \frac{d}{dt}(\text{upper limit}) \times \text{Integrand}\Big|_{\substack{\text{upper}\\\text{limit}}} - \frac{d}{dt}(\text{lower lim}) \times \text{Integrand}\Big|_{\substack{\text{lower}\\\text{limit}}}$$

Apply Leibniz rule

$$+ \int_s^t \frac{\partial}{\partial t}\left(\text{Integrand}(t)\right) d\tau$$

$$= 1 . \overset{I}{\cancel{\Phi(t,t)}} B(t) B'(t) \overset{I}{\cancel{\Phi'(t,t)}} - \overset{0}{\cancel{\frac{ds}{dt}}} \times (\cdot\,\cdot)$$

$$+ \int_s^t \left\{ \left(\overset{A(t)\Phi(t,\tau)}{\cancel{\frac{\partial \Phi(t,\tau)}{\partial t}}} \right) B(\tau) B'(\tau) \Phi'(t,\tau) + \Phi(t,\tau) B(\tau) B'(\tau) \left(\overset{\Phi^T A'(t)}{\cancel{\frac{\partial}{\partial t} \Phi(t,\tau)}} \right)' \right\} d\tau$$

product rule of differentiation

$$= B(t)\,B'(t) + A(t)\,M(t,s) + M(t,s)\,(A(t))'$$

Controllability Gramian for the LTI case $(A(t), B(t)) \equiv (A, B)$

(i.e.) system matrices are constant

Then

$$M(s,t) = \int_s^t e^{A(t-\tau)}\, B\,B'\, e^{A'(t-\tau)}\, d\tau$$

(Here " $'$ " (prime) denotes matrix transpose)

In the LTI case, Lyapunov ODE becomes:

$$\boxed{\dot{M}(t) = A\,M(t) + M(t)A^T + B\,B'}$$

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In the LTI case, the following ③ sentences are equivalent

(I) System (A, B) is controllable

(II) The controllability Gramian M is non-singular (hence > 0)

(III) The Kalman rank condition holds:

$$\text{rank}\left[B \,|\, AB \,|\, A^2 B \,|\, \dots \,|\, A^{n-1} B\right] = n$$

where n is the dimension of state space.

Remark:

① In the LTI case, we do not need to talk about the time interval $[0, 1]$ (or, $[0, t_{final}]$)

② The analogue of (III) for the LTV case is rather complicated, we will NOT discuss that