$\boxed{t_f \equiv T}$

Lecture #6

Cost function/objective

$$\min_{u(\cdot)} \boxed{J(u) = \underbrace{\phi(x(T), T)}_{\text{terminal cost}} + \int_{t_0=0}^{t_f=T} \underbrace{L(\underline{x}, \underline{u}, t)}_{\text{"cost-to-go"}} dt}$$

Bolza form (general form) for OCP

Lagrange form (when $\phi(\cdot) \equiv 0$) (only cost-to-go) $\Longleftrightarrow$ equivalent

Mayer form ($L \equiv 0$) (only terminal cost)

equivalent

Equivalence betⁿ. forms :

$$\boxed{\text{Mayer form} \longrightarrow \text{Lagrange form}}$$

$$\phi(x(T),T)$$

$$= \phi(t_0, x_0) + \int_{t_0}^{t_f} \frac{d}{dt}\phi(t, x(t))\,dt$$

$$= \underbrace{\phi(t_0, x_0)}_{\text{indep. of } u(\cdot)} + \int_{t_0}^{t_f}\left\{\frac{\partial \phi}{\partial t} + \langle \nabla_{\underline{x}}\phi, \underline{f}\rangle\right\}dt$$

$$\underbrace{\arg\min_{u(\cdot)} \phi(x(T),T)}_{\text{Mayer form.}} = \underbrace{\arg\min_{u(\cdot)} \int_{t_0}^{t_f}\underbrace{\left\{\frac{\partial \phi}{\partial t} + \langle \nabla_{\underline{x}}\phi, \underline{f}\rangle\right\}}_{L}dt}_{\text{Lagrange form}}$$

∎

$\boxed{\text{Lagrange form } (L) \longrightarrow \text{Mayer form } (\phi)}$

Introduce extra state variable $\tilde{x}$ as

$$\dot{\tilde{x}} = L(t, \underline{x}, \underline{u})$$

$\tilde{x}(t_0)$ = arbitrary constant
(only changes the cost by an additive constant, does NOT change argmin)

$$\therefore \int_{t_0}^{t_f} L(t, \underline{x}, \underline{u})\, dt = \tilde{x}(t_f)$$

by construction

(plays the role of ϕ)

$\blacksquare$

First order Necessary Conditions for Optimality (1958)

OPT	CoV	OCP (Pontryagin & Boltyanskii)
KKT condition	EL equation	Now

Hamiltonian $H(\underline{x}(t), \underline{u}(t), \underline{\lambda}(t), t)$ (scalar)

$$:= L(\underline{x}(t), \underline{u}(t), t) + \underbrace{\underline{\lambda}^T(t)\, f(\underline{x}(t), \underline{u}(t), t)}$$

① State eq^n:

$$\underline{\dot{x}}(t) = \frac{\partial H}{\partial \underline{\lambda}} = f(\underline{x}, \underline{u}(t), t), \quad \underline{x}(t) \in \mathbb{R}^n$$

$$\underline{x}(0) = \underline{x}_0 \text{ (given)}$$

② Co-state eq^n:

$$\underline{\dot{\lambda}}(t) = -\frac{\partial H}{\partial \underline{x}}, \quad \underline{\lambda}(t) \in \mathbb{R}^n$$

(time-varying) Lagrange multiplier/trajectories

We call $\underline{\lambda}(t)$ as co-state

③ Pontryagin's Maximum Principle (PMP) :

$$\underset{m\times 1}{\underline{0}} = \frac{\partial H}{\partial \underline{u}} \, , \qquad \underline{u} \in \mathbb{R}^m$$

④ Transversality Condition :

$$\left(\nabla_{\underline{x}} \phi + \left(\nabla_{\underline{x}} \underline{\psi}\right)^T \underline{\nu} - \underline{\lambda} \right)^T \Bigg|_{t = t_f \equiv T} d\underbrace{\underline{x}(T)}_{\text{final state}} +$$

$$\left(\frac{\partial \phi}{\partial t} + \left(\frac{\partial \underline{\psi}}{\partial t}\right)^T \underline{\nu} + H \right) \Bigg|_{t = T} d\underbrace{T}_{\text{final time}} = 0$$

$\underline{\nu}$: Lagrange multiplier vector but constant vector

Hamiltonian H :

$$\frac{d}{dt} H(\underline{x}, \underline{u}, \underline{\lambda}, t)$$

$$\underset{\text{(chain rule)}}{=} \frac{\partial H}{\partial t} + (\nabla_{\underline{x}} H)^T \cancelto{\underline{f}}{\dot{\underline{x}}} + (\nabla_{\underline{u}} H)^T \dot{\underline{u}} + (\dot{\underline{\lambda}})^T f$$

$$= \frac{\partial H}{\partial t} + \Big(\underbrace{\cancel{\nabla_{\underline{u}} H}}_{\substack{0 \\ \text{(from PMP)} \\ \text{(condition 3)}}}\Big)^T \dot{\underline{u}} + \Big(\underbrace{\nabla_{\underline{x}} H + \dot{\underline{\lambda}}}_{\substack{\equiv 0 \\ \text{(condition 2)}}}\Big)^T \underline{f}$$

$$= \frac{\partial H}{\partial t}$$

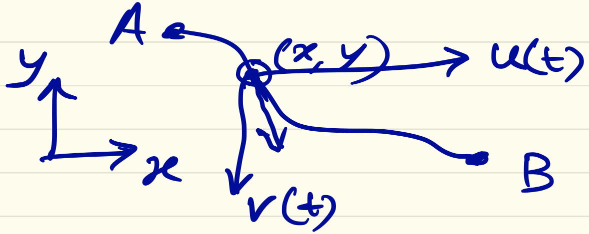
$\Rightarrow H^*$ is constant if the OCP is <u>time-invariant</u>
(i.e., neither f nor L depends explicitly on t)

Proof of (1)-(4) : summary/outline : Book by Liberzon.

Actual proof : Agrachev- Sachkov ✓

we will not prove it

Example 1 : (Shortest planar path is straight line, revisited)



① State eq^n:

$$\dot{x} = u$$
$$\dot{y} = v$$

2 states, 2 controls

$$\int_A^B \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1+\frac{v^2}{u^2}}\, dt$$

$$\phi = 0$$

Hamiltonian :

Also $\psi = 0$

$$H = \sqrt{1 + \frac{v^2}{u^2}} + \underline{\lambda}^T \underline{f}$$

$$\boxed{\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \end{aligned}} \quad \text{--- (1)}$$

$$= \sqrt{1 + \frac{v^2}{u^2}} + \lambda_1(t) u + \lambda_2(t) v$$

<u>Apply co-state eq^n : (condition (2))</u>

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0 \Rightarrow \lambda_1(t) = c_1$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = 0 \Rightarrow \lambda_2(t) = c_2$$

<u>Apply PMP (condition (3))</u>

$$0 = \frac{\partial H}{\partial u} = \frac{v^2/u^2}{\sqrt{u^2+v^2}} + \cancel{\lambda_1} \, c_1$$

$$0 = \frac{\partial H}{\partial v} = \frac{v/u}{\sqrt{u^2+v^2}} + \cancel{\lambda_2} \, c_2$$

2 eq^ns in 2 unknowns

$\therefore$ $u = K_1$ (some constant that is nonlinear fn. of other constants c_1, c_2)

$v = K_2$ (another constant depends on c_1 & c_2)

<u>But state eqn:</u>

$$\left.\begin{aligned} \dot{x} = u = K_1 &\Rightarrow x^* = K_1 t + \tilde{K}_1 \\ \dot{y} = v = K_2 &\Rightarrow y^* = K_2 t + \tilde{K}_2 \end{aligned}\right\}$$

eliminate t

$\Downarrow$

$\therefore$ <u>st. line is optimal.</u>

$$y^* = l_1 x^* + l_2$$

$$l_1 = K_2/K_1 ,$$

$$l_2 = \tilde{K}_2 - \frac{K_2}{K_1}\tilde{K}_1$$

Apply $\left.\begin{aligned} (x_1, y_1) &\equiv A \\ (x_2, y_2) &\equiv B \end{aligned}\right\}$ to determine the constants

$$H^* = H(x^*, u^*, \lambda^*, t)$$

$$= \sqrt{1 + \frac{k_2^2}{k_1^2}} + c_1 k_1 + c_2 k_2$$

$= \text{constant (verified.)}$ ∎

Example 2: (Temperature (θ) control in a room)

Newton's Law of heating/cooling:

$$\dot{\theta} = -\alpha(\theta - \underbrace{\theta_a}_{\text{constant}}) + \beta u,$$

α, β = constants

θ_a = ambient temperature = constant.

Let $x(t) := \theta(t) - \theta_a$

$$\Rightarrow \boxed{\dot{x} = -\alpha x + \beta u}, \quad x(0) = x_0 \text{ known}.$$

u = control

$$\underset{u(\cdot)}{\text{minimize}} \int_0^T \underbrace{\frac{1}{2}(u(t))^2}_{\text{Energy.}} dt \,, \quad T = \text{final time } \text{fixed}$$

$$x(0) = x_0 \text{ (known)}$$

$$H = L + \lambda f$$

$$= \frac{1}{2}u^2 + \lambda(t)(-\alpha x + \beta u)$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = -\alpha x + \beta u \leftarrow \text{state eqn}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = +\alpha \lambda(t) \leftarrow \text{costate eqn}$$

$$0 = \frac{\partial H}{\partial u} = u + \lambda(t)\beta$$

$$\Rightarrow \boxed{u^*(t) = -\beta \lambda(t)}$$

For now, let's pretend that $\underline{\underline{\lambda(T)}}$ is known.

Then $\dot{\lambda} = +\alpha\lambda$

$\Rightarrow$ (integrate back in time)

$$\boxed{\lambda(t) = e^{-\alpha(T-t)}\lambda(T)}$$

Then $\dot{x} = -\alpha x(t) + \beta \cdot u(t)$

$$= -\alpha x(t) + \beta \cdot (-\beta\lambda(t))$$

$$= -\alpha x - \beta^2\lambda(t)$$

$$= -\alpha x - \beta^2 e^{-\alpha(T-t)}\lambda(T)$$

$$\Rightarrow x(t) = x_0 e^{-\alpha t} - \frac{\beta^2}{\alpha}\lambda(T) e^{-\alpha T}\sinh(\alpha t)$$

$\boxed{\text{Case I}}$ Fixed $x(T)$.

Give T is also fixed.

$$\left.\begin{aligned} dx(T) &= 0 \\ dT &= 0 \end{aligned}\right\} \quad x(0) = 0$$

$\therefore$ Condition (4) gives us nothing new.

Then the only strategy possible to compute $\lambda(T)$ is:

$$\underbrace{x(t)\Big|_{t=T}}_{\substack{x(T)\\(\text{given})}} = \underbrace{x(0)}_{0} e^{-\alpha T} - \frac{\beta^2}{2\alpha}\lambda(T)\left\{1 - e^{-2\alpha T}\right\}$$

$$\Rightarrow \lambda(T) = fn(\underbrace{x(T)}_{\text{Known}}, \underbrace{T}_{\text{Known}})$$

$\therefore$ solved.

$\boxed{\text{Case II}}$ Free terminal state $x(T)$

but we need to put $\phi(x(T),T)$

$$\underset{\uparrow \text{ cost}}{J} = \underbrace{\frac{1}{2}\,\underset{\substack{\uparrow \\ \text{some positive constant.}}}{s}\,(x(T)-10)^2}_{\phi(x(T))} + \frac{1}{2}\int_0^T u^2\,dt$$

Then we need condition (4).

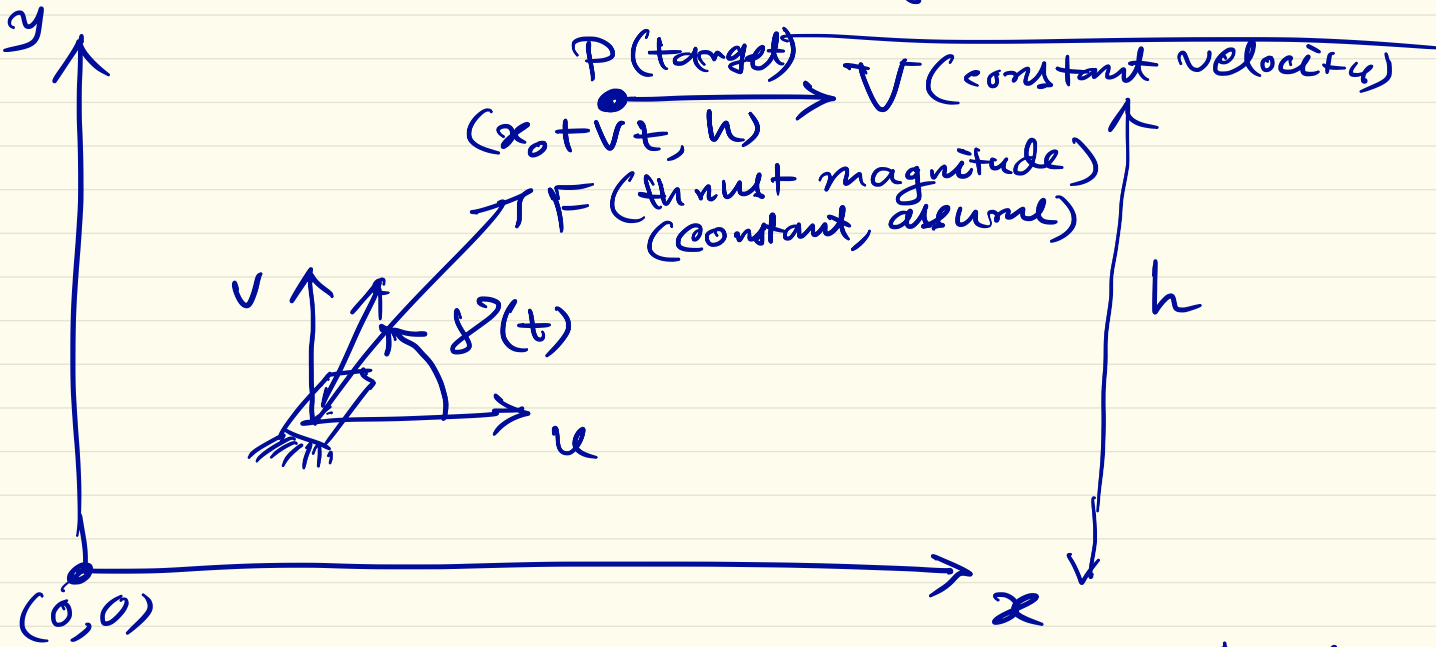
$$dT = 0$$
$$dx(T) \neq 0$$

$\therefore$ coeff. of $dx(T)$, must be $= 0$.

That gives $\lambda(T) = s\,(x(T)-10)$.

Example 3:

Thrust angle Programming: Intercepting moving target by a missile



Intercept problem vs. Rendezvous Problem

(Missile problem) (Moon landing)

State vector of the missible $(\underline{x}) = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} \in \mathbb{R}^4$

$a := \dfrac{F}{m}$ (known thrust accelⁿ)

Target P has initial posⁿ x_0 (if moves @ const. altitude h)

Control: thrust angle $\gamma(t)$.

Controlled dynamics (state ODE)

---

state $\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$

$$\left.\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= a\cos\gamma \\ \dot{v} &= a\sin\gamma \end{aligned}\right\} \quad \begin{aligned} &\text{I.C.} \\ &x(0) = 0 \\ &y(0) = 0 \\ &u(0) = 0 \\ &v(0) = 0 \end{aligned}$$