Lecture # 5

The following choice will work:

$$L = \underbrace{T(\dot{\underline{q}})}_{\text{Kinetic energy}} - \underbrace{V(\underline{q})}_{\text{Potential energy}}$$

Kinetic energy $= \frac{1}{2} m \|\dot{\underline{q}}\|^2$

$$= \frac{1}{2} m \langle \dot{\underline{q}}, \dot{\underline{q}} \rangle$$

Cartesian co-ordinate: $\underline{q} = (x, y, z)^T$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Cylindrical:

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

Spherical

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, (\dot{\phi})^2)$$

CoV Problem:

$$\min_{u} I(u) = \int_{t=t_0}^{t=t_f} L(t, \underline{q}, \dot{\underline{q}}) \, dt$$

$$\left\{ T(\dot{\underline{q}}) - V(\underline{q}) \right\}$$

EL eqⁿ:

$$L_{\underline{q}} = \frac{d}{dt} L_{\dot{\underline{q}}} \xRightarrow{\text{Substitute } L = T(\dot{\underline{q}}) - V(\underline{q})} -\frac{\partial V}{\partial \underline{q}} = \frac{d}{dt}(m \dot{\underline{q}})$$

$$\Rightarrow -\frac{\partial V}{\partial \underline{q}} = m \ddot{\underline{q}} \quad \text{Newton's Law}$$

The identification that $L = T - V$, is called
Hamilton's principle of least action.

- $L = T - V$, and $T = \frac{1}{2} m \langle \dot{\underline{q}}, \dot{\underline{q}} \rangle$
- when $V = 0$, $L = T(\dot{\underline{q}})$
$$= \frac{1}{2} m \langle \dot{\underline{q}}, \dot{\underline{q}} \rangle$$

Then CoV problem:
$$\min_{\underline{q}(t)} I(\underline{q}) = \int_{t=t_0}^{t=t_f} \frac{1}{2} m \langle \dot{\underline{q}}, \dot{\underline{q}} \rangle dt$$
$$= \frac{1}{2} m \int_{t=t_0}^{t=t_f} \langle \dot{\underline{q}}, \dot{\underline{q}} \rangle dt$$

curves in $\mathbb{R}^n$

minimizers are straight line

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 \Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} \quad (\text{in } 2D)$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt$$

- If $V \neq 0$, then $\min_{\underline{q}(t)} I(\underline{q}) = \int_{t=t_0}^{t=t_f} \{T(\dot{\underline{q}}) - V(\underline{q})\} dt$

minimizing curves/paths/trajectories are "generalized straight lines" called "geodesics".
w.r.t. metric that depends on potential energy $V(\underline{q})$.

- Newton's Law $\iff$ EL equation
(Pointwise statement) minimizing path (Global statement)

• If the force field is conservative ($F(\underline{q}) = -\frac{\partial V}{\partial \underline{q}}$)

then neither T nor V, depends explicitly on t (time)

$$L(\underline{q}, \dot{\underline{q}}) = T(\underline{q}, \dot{\underline{q}}) - V(\underline{q}, \dot{\underline{q}})$$

$\therefore$ we can apply Beltrami identity:

$$L - \left\langle \dot{\underline{q}}, \frac{\partial L}{\partial \dot{\underline{q}}} \right\rangle = \text{constant}$$

$\Leftrightarrow$ (its negative)

$$\underbrace{\left\langle \dot{\underline{q}}, \frac{\partial L}{\partial \dot{\underline{q}}} \right\rangle - L}_{\text{we call this "Hamiltonian" } (H)} = \text{constant}$$

Hamilton's canonical equations:

Let $\underline{p} := L_{\underline{\dot{q}}}(t, \underline{q}, \underline{\dot{q}})$

We can think of the vector $\underline{p}$ as fn. of t, associated with a given path/trajectory $\underline{q}(t)$.

We defined: $\underbrace{H(t, \underline{q}, \underline{\dot{q}}, \underline{p})}_{\substack{\text{written as a fn of} \\ \text{4 variables but}}} = \langle \underline{\dot{q}}, \underline{p} \rangle - L(t, \underline{q}, \underline{\dot{q}})$

becomes a fn of "t" alone when evaluated along a particular trajectory $\underline{q}(t)$.

- We call "$\underline{q}$" & "$\underline{p}$" as $\boxed{\text{"canonical variables"}}$
- Suppose $\underline{q}(t)$ satisfies EL equation.
- We can write ODEs for $\underline{q}(t)$ & $\underline{p}(t)$ along the soln of EL eqn., in terms of H as:

$$\frac{d}{dt}\underline{q} = \frac{\partial}{\partial \underline{p}} H(t, \underline{q}(t), \underline{\dot{q}}(t)), \quad \frac{d}{dt}\underline{p} = \frac{d}{dt} L_{\underline{\dot{q}}} \overset{\text{EL eqn}}{=} L_{\underline{q}} = -\frac{\partial H}{\partial \underline{q}}$$

$\therefore$ we have the canonical ODEs :

$$\boxed{\begin{aligned} \frac{d\underline{q}}{dt} &= \frac{\partial}{\partial \underline{p}} H \\ \frac{d\underline{p}}{dt} &= - \frac{\partial}{\partial \underline{q}} H \end{aligned}}$$

This is simply a reformulation of EL eqn. in terms of Hamiltonian H (proposed by Hamilton, 1835)

• Mathematically "What is Hamiltonian"?

Ans. H is the dual/convex conjugate/Legendre-Fenchel conjugate of $L(\underline{q}, \dot{\underline{q}})$ w.r.t. vector $\dot{\underline{q}}$

Aside: convex conjugate/Legendre-Fenchel conjugate of a function $f(\underline{x})$ is (where $f: \mathbb{R}^n \mapsto \mathbb{R}$)

$$g(\underline{y}) = \sup_{\underline{x} \in \mathbb{R}^n} \{ \langle \underline{y}, \underline{x} \rangle - f(\underline{x}) \},$$

shorthand: $f^* = g$

$$H = \left(L(\underline{q}, \dot{\underline{q}}) \right)^* = \underbrace{\sup_{\dot{\underline{q}} \in \mathbb{R}^n} \left\{ \langle \underline{\xi}, \dot{\underline{q}} \rangle - L(\underline{q}, \dot{\underline{q}}) \right\}}_{\text{unconstrained minimization}}$$

So, the maximizing $\dot{\underline{q}}$ solves: $\frac{\partial}{\partial \dot{\underline{q}}} \left\{ \langle \underline{\xi}, \dot{\underline{q}} \rangle - L(\underline{q}, \dot{\underline{q}}) \right\} = 0$

$\Rightarrow \underline{\xi} - \frac{\partial L}{\partial \dot{\underline{q}}} = 0$

∴ After the maximization:

$$H = \left(L(\underline{q}, \underline{\dot{q}})\right)^* = \underbrace{\left\langle \frac{\partial L}{\partial \underline{\dot{q}}}, \underline{\dot{q}} \right\rangle - L(\underline{q}, \underline{\dot{q}})}$$

$$= \text{constant}$$

(from Beltrami)
(if applicable)

Recall:

$$\frac{\partial L}{\partial \underline{\dot{q}}} = \underline{\xi}$$

i.e., H may not be constant if we cannot apply Beltrami (e.g. if L has explicit dependence on time "t")

<u>Physically</u>: (If conservative force field)

$$H = \left\langle \underline{\dot{q}}, \frac{\partial L}{\partial \underline{\dot{q}}} \right\rangle - L$$

$$= \langle \underline{\dot{q}}, m\underline{\dot{q}} \rangle - \left\{ \frac{1}{2} m \langle \underline{\dot{q}}, \underline{\dot{q}} \rangle - V(\underline{q}) \right\}$$

$$L = T - V = \frac{1}{2} m \langle \underline{\dot{q}}, \underline{\dot{q}} \rangle - V(\underline{q})$$

$$\Rightarrow \quad H = \frac{1}{2} m \langle \dot{\underline{q}}, \dot{\underline{q}} \rangle - (-V(\underline{q}))$$

$$= \underbrace{\frac{1}{2} m \langle \dot{\underline{q}}, \dot{\underline{q}} \rangle}_{\substack{\text{III} \\ T(\dot{\underline{q}})}} + \underbrace{V(\underline{q})}_{}$$

$$= \text{Kinetic energy} + \text{Potential energy}$$

$$= \text{Total Energy } (E)$$

If conservative force field, then

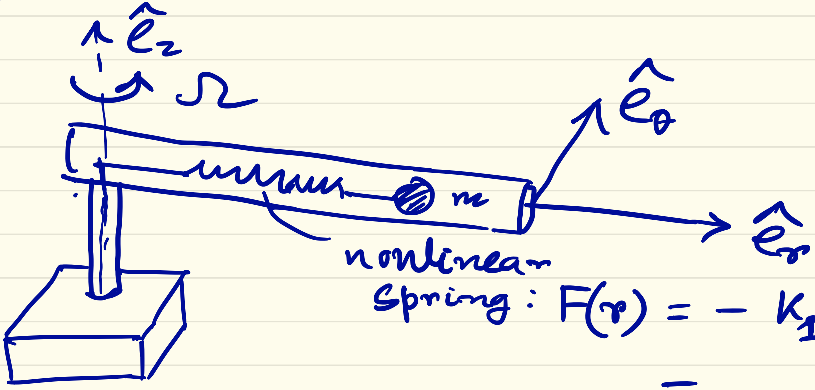
$$H = T + V = E = \text{constant}$$

$$L = T - V \text{ (even if non-conservative)}.$$

If not, then $H = E$ may not hold

also, one or none of H & E may be constant of motion.

Example (EL equation for ball in rotating tube)



spring: $F(r) = -k_1 r - k_2 r^3 = -\dfrac{\partial V}{\partial r}$ $\Big\}$ $V(r) = \frac{1}{2}k_1 r^2 + \frac{1}{4}k_2 r^4$

position vector: $\underline{r}(t) = r\hat{e}_r$

velocity vector: $\dot{\underline{r}}(t) = \dfrac{d}{dt}\underline{r}(t) \underset{\uparrow}{=} \dot{r}\hat{e}_r + \underbrace{\underline{\omega}}_{\{(\Omega\hat{e}_z)\times(r\hat{e}_r)\}} \times \underline{r}$

(Transport theorem)

$= \dot{r}\hat{e}_r + r\Omega\underbrace{(\hat{e}_z \times \hat{e}_r)}_{\hat{e}_\theta}$

$= \boxed{\dot{r}\hat{e}_r + r\Omega\hat{e}_\theta}$

$T(\dot{\underline{r}})$
$= \frac{1}{2}m\langle \dot{\underline{r}}, \dot{\underline{r}} \rangle$
$= \frac{1}{2}m(\dot{r}^2 + \Omega^2 r^2)$

$$L = T - V$$

$$= \frac{1}{2} m(\dot{r}^2 + \Omega^2 r^2) - \left\{\frac{1}{2}k_1 r^2 + \frac{1}{4}\Omega^2 r^2\right\}$$

$$= \frac{1}{2} m\dot{r}^2 + \left(\frac{m\Omega^2}{2} - \frac{k_1}{2}\right) r^2 - \frac{1}{4}k_2 r^4$$

Now apply EL eqⁿ:

$$\frac{\partial L}{\partial \underline{q}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \underline{\dot{q}}}\right)$$

Here, $\underline{q} = \underline{r}$

$$\Rightarrow (m\Omega^2 - k_1) r - k_2 r^3 = m\ddot{r}$$

$$\Leftrightarrow \boxed{m\ddot{r} + (k_1 - m\Omega^2) r + k_2 r^3 = 0}$$

Now, total energy

$$E = T + V = \frac{1}{2} m\dot{r}^2 + \frac{k_1}{2} r^2 + \frac{k_2}{4} r^4$$

But $H = \frac{\partial L}{\partial \dot{r}}\dot{r} - L = \frac{1}{2} m\dot{r}^2 + \left(\frac{k_1}{2} - \frac{m\Omega^2}{2}\right) r^2 + \frac{k_2 r^4}{4}$

Clearly, $H \neq E$.

In fact, by direct differentiation,

$$\frac{d}{dt}H = \dot{r}\underbrace{\left\{m\ddot{r} + (K_1 - m\Omega^2)r + K_2 r^3\right\}}_{=0 \text{ (from EL eqn)}}$$

$$= 0$$

$\therefore$ H is a constant of motion

$$\frac{dE}{dt} = \dot{r}\underbrace{\left\{m\ddot{r} + K_1 r + K_2 r^3\right\}}$$

$$= \dot{r}\, m\Omega^2 r = m\Omega^2 r\dot{r} \neq 0$$

$\therefore$ Total Energy is not conserved in this system.

we mentioned about pointwise equality constraint.

If $M(x, u) = 0$ (i.e.) no dependence on ∇u)

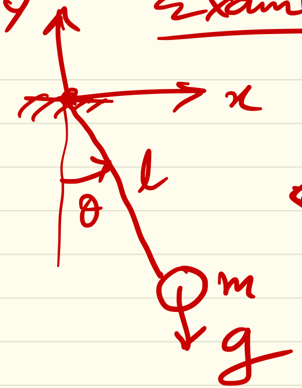
then this pointwise equality constraint is called "holonomic constraint" ($\Leftrightarrow$ system is overparameterized)

can be thought of as constraint surface $(\underline{x}, u(\underline{x}))$

$\Rightarrow$ Instead of going the Lagrange multiplier route, we can actually make it an unconstrained CoV prolem, by eliminating variables.

Example next pg.

Example: (Simple pendulum)



$$M(x,y) = x^2+y^2-l^2 = 0$$

$$\Leftrightarrow M(r,\theta) = r^2 - l^2 = 0$$

$$\Leftrightarrow r = l = \text{constant}$$

$$\Leftrightarrow \dot{r} = 0$$

Can avoid Lagrange multipliers

Now, kinetic energy:

unconstrained CoV

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (\cancelto{0}{\dot{r}^2} + r^2 \dot{\theta}^2)$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = mgl(1 - \cos\theta)$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(\cos\theta - 1)$$

EL eqⁿ: $\boxed{\frac{\partial L}{\partial \theta} = \frac{d}{dt} L_{\dot{\theta}}} \Leftrightarrow \boxed{-mgl\sin(\theta) = \frac{d}{dt}(ml^2\dot{\theta})}$

$$-\cancel{m}gl\sin(\theta) = \cancel{m}l^2\ddot{\theta}$$

$$\Leftrightarrow \boxed{\left(-\frac{g}{l}\right)\sin\theta = \ddot{\theta}}$$

From CoV to OCP (Optimal control Problem):

Template:

$$\min_{u(\cdot)\in\mathcal{U}([t_0,t_f])} J(\underline{u}) := \underbrace{\phi(\underline{x}(t_f), t_f)}_{\text{terminal cost}} + \int_{t_0}^{t_f} \overbrace{\underbrace{L(\underline{x}, \underline{u}, t)}_{\text{Lagrangian}}}^{\text{"cost-to-go"}}\, dt$$

s.t. ① $\underset{n\times 1}{\dot{\underline{x}}} = \underset{n\times 1}{\underline{f}}(\underline{x}, \underline{u}, t)$, $\underline{x}(0) = \underline{x}_0$ (given initial condition)

controlled dynamics

② $\underline{\psi}(\underline{x}(t_f), t_f) = 0$ } Terminal constraint

we call: $\underline{x}(t)$ as state vector, $\underline{x}: [t_0, t_f] \mapsto \mathbb{R}^n$

$\underline{u}(t)$ as control " , $\underline{u}: [t_0, t_f] \mapsto \mathbb{R}^m$

Final time t_f may be "free" OR "fixed"