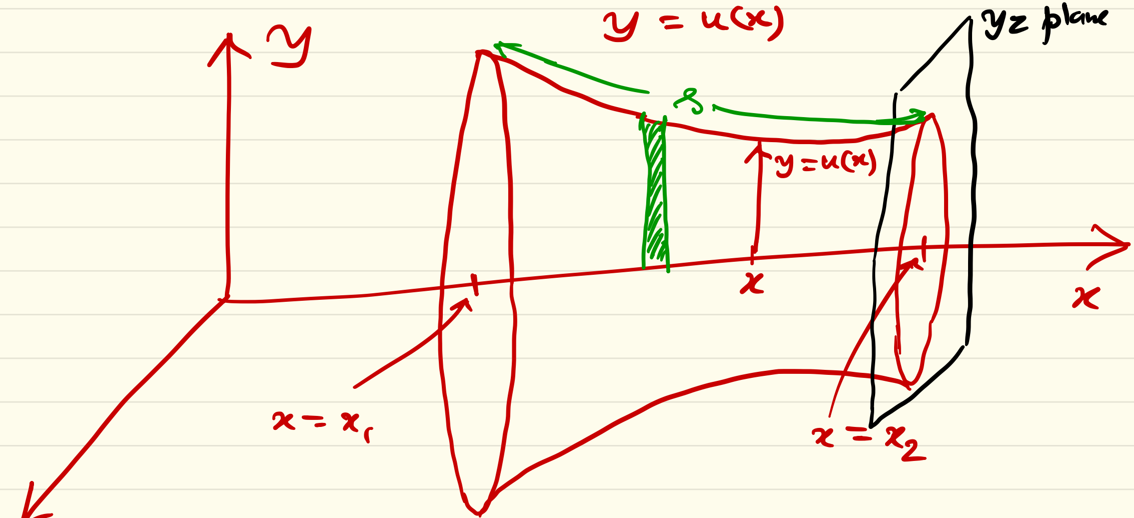
Example

Lecture #4

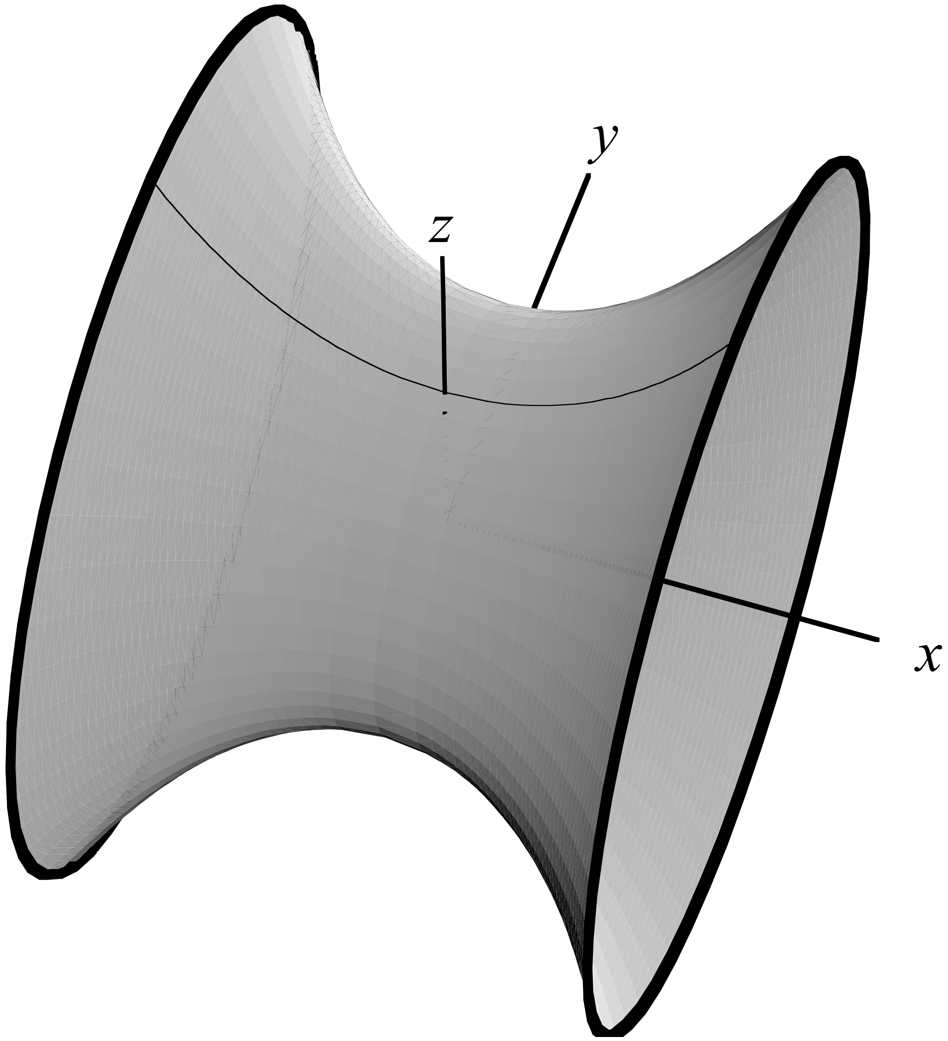
Minimal Surface Problem
(Shape of co-axial Soap Film)



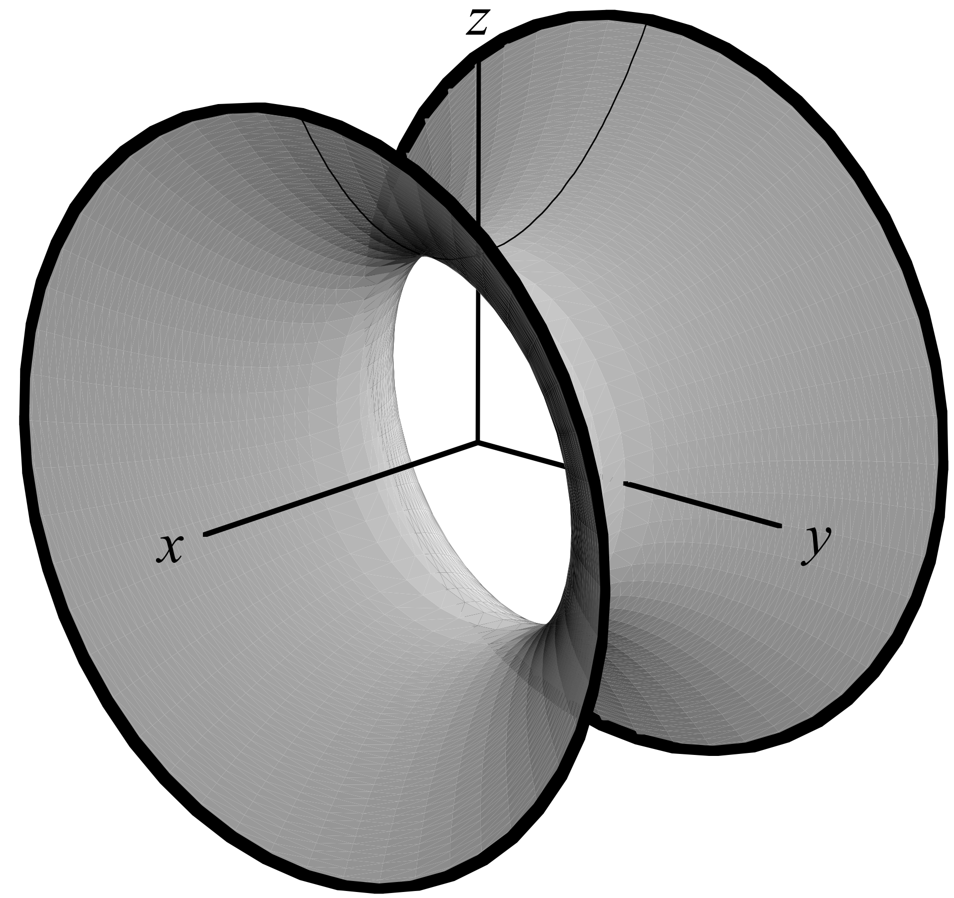
Minimum energy configuration :

$$\text{Energy due to surface tension} = \underbrace{\left(\begin{array}{c}\text{Surface} \\ \text{tension coefficient}\end{array}\right)}_{\substack{\sigma \\ \text{(some constant)}}} \times \underbrace{\left(\begin{array}{c}\text{Surface} \\ \text{area}\end{array}\right)}_{\substack{\text{Area of revolution} \\ \text{of curve } y = u(x) \\ \text{(due to symmetry)}}}$$

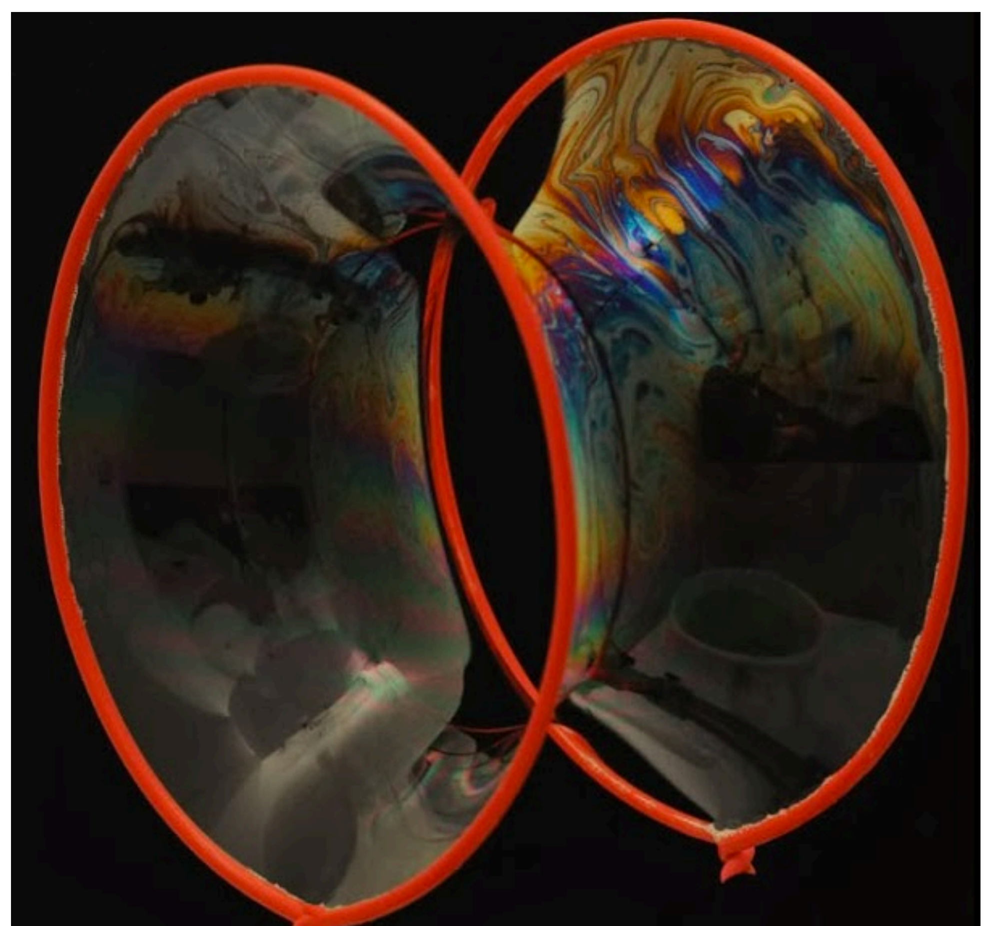
$$\therefore I(u) = \int_{x_1}^{x_2} d(\text{Energy}) = \sigma \int_{x_1}^{x_2} y \, ds = \sigma \int_{x_1}^{x_2} u \sqrt{1+(u')^2}\, dx$$



y
z
x



z
x
y



$\therefore$ CoV Problem : $\displaystyle\min_{u \in C^1(\Omega)} \int_{x_1}^{x_2} u\sqrt{1+(u')^2}\,dx$

s.t. $u(x_1) = y_1$, $u(x_2) = y_2$

Since Lagrangian $L = u\sqrt{1+(u')^2}$ is indep. of x.

$\therefore$ Beltrami identity :

$$L - u' \frac{\partial L}{\partial u'} = c \leftarrow \text{constant}$$

$$\Rightarrow u\sqrt{1+(u')^2} - u' \cdot \frac{u \cdot \cancel{2}u'}{\cancel{2}\sqrt{1+(u')^2}} = c$$

$$\Rightarrow \frac{u + \cancel{u(u')^2} - \cancel{u(u')^2}}{\sqrt{1+(u')^2}} = c$$

$$\Rightarrow \frac{u}{\sqrt{1+(u')^2}} = c$$

$$\Rightarrow \frac{du}{dx} = \sqrt{\frac{u^2}{c^2} - 1}$$

substitute :

$u = c\cosh(z)$

$\Leftrightarrow z = \cosh^{-1}\left(\frac{u}{c}\right)$

$\therefore \frac{du}{dx} = \sinh(z)\frac{dz}{dx}$

$\Rightarrow c\sinh(z)\frac{dz}{dx} = \sinh(z)$

$\Rightarrow z = \frac{x}{c}$

$$\Rightarrow \underbrace{u(x) = c \cosh\left(\frac{x}{c}\right)}_{\text{Catenary (curve)}}$$

Surface of revolution is called "catenoid".

B.C.

$$y_1 = u(x_1) = c\cosh\left(\frac{x_1}{c}\right)$$
$$y_2 = u(x_2) = c\cosh\left(\frac{x_2}{c}\right)$$

Given (x_1, y_1) & (x_2, y_2), does there exist a "c" that solve both these equations?

Answer requires numerical simulation.

Slightly tractable:

w.l.o.g. $x_1 = -L, \quad x_2 = +L$

$y_1 = y_2 = r$

Then, we need to solve :

$$\frac{r}{c} = \cosh\left(\frac{L}{c}\right),$$

$$\Longleftrightarrow \boxed{K r = \cosh(KL)} \quad \text{where } K := 1/c.$$

Given $r, L > 0$ solve for K.

There could be 0, 1, 2 roots.

Can show that this eq^{n} may have 2 roots K_1, K_2 satisfying the bound :

$$\frac{1}{r} \leq K_1 < K^* < K_2 \leq \frac{2r}{L^2}$$

where $K^* := \frac{1}{L} \sinh^{-1}(r/L)$

If $\cosh(K^* L) = K^* r$ then $K_1 = K_2 = K^*$ (unique sol^{n})

If $\cosh(K^* L) < K^* r$ then 2 sol^{n}s satisfying the bound

If $\cosh(\kappa^* L) > \kappa^* r$ then no sol^{n}.

End of example

Interpretation of EL eq^{n} as Gradient Descent

OPT problem : $\min_{\underline{x} \in \mathbb{R}^n} u(\underline{x})$,

$$\nabla_{\underline{x}} u = 0$$

CoV : Interpret $\frac{\partial L}{\partial u} - \nabla \cdot \left(\frac{\partial L}{\partial \nabla u} \right) = 0$

as $\underbrace{\nabla}_{??} I(u) = 0$

Note that in finite dim. OPT, $u : \mathbb{R}^n \mapsto \mathbb{R}$

$$\nabla u = (u_{x_1}(\underline{x}), \ldots, u_{x_n}(\underline{x}))$$

By chain rule,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} u(\underline{x} + \epsilon \underline{v}) = \langle \nabla u, \underline{v} \rangle \quad \forall\, \underline{v} \text{ in } \mathbb{R}^n$$

$\therefore$ Inner product defines gradient

$\therefore\ \underline{w} = \nabla u(\underline{x})$ is the unique vector satisfying

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} u(\underline{x}+\epsilon\underline{v}) = \langle \underline{w}, \underline{v} \rangle \quad \forall\ \underline{v} \text{ in } \mathbb{R}^n.$$

In the EL proof, we showed that

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(u+\epsilon\phi) = \underbrace{\left\langle \frac{\partial L}{\partial u} - \nabla\cdot\frac{\partial L}{\partial \nabla u}, \phi \right\rangle_{L^2(\Omega)}}_{\int_\Omega \left(\frac{\partial L}{\partial u} - \nabla\cdot\frac{\partial L}{\partial \nabla u}\right)\phi\, d\underline{x}}$$

$\boxed{\text{Here } \langle *, \$ \rangle = \int_\Omega * \$ \, dx}$

(i.e) L^2 inner product plays the role of vectorial dot product in finite dim.

If we define $\nabla I(u) := \frac{\partial L}{\partial u} - \nabla\cdot\frac{\partial L}{\partial \nabla u}$

then $\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(u+\epsilon\phi) = \langle \nabla I(u), \phi \rangle_{L^2(\Omega)}$

Notice that the gradient of the functional $I(u)$ depends on the choice of <u>inner product</u>.

<u>Implication:</u>

Numerical simulation of this is gradient descent (*)

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\nabla_u I(u) & \text{for } (\underline{x},t) \in \Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for } (\underline{x},t) \in \Omega \times \{0\} \end{cases}$$

Stationary pt. $\dfrac{\partial u}{\partial t} = 0$ is the critical pt. for EL eq^n

<u>Claim:</u> To see gradient descent decreases I, suppose $u(x,t)$ solves (*)

$$\frac{d}{dt} I(u) = \int_\Omega \frac{d}{dt} L(\underline{x}, u, \nabla u)\, dx$$

$$= \int_{\Omega} \Big\{ \frac{\partial L}{\partial u} \frac{\partial u}{\partial t} + \underbrace{\frac{\partial L}{\partial \nabla u} \frac{\partial \nabla u}{\partial t}}_{\text{Integration by parts}} \Big\} \, dx \quad (\text{chain rule})$$

$$= \int_{\Omega} \Big\{ \frac{\partial L}{\partial u} - \nabla \cdot \Big(\frac{\partial L}{\partial \nabla u} \Big) \Big\} \frac{\partial u}{\partial t} \, dx$$

$$= \Big\langle \nabla I(u), \underset{\parallel}{\frac{\partial u}{\partial t}} \Big\rangle_{L^2(\Omega)}$$

$$= \langle \nabla I(u), -\nabla I(u) \rangle_{L^2(\Omega)}$$

$$= - \| \nabla I(u) \|^2_{L^2(\Omega)}$$

$$\leq 0$$

Summary: Gradient descent PDE is

$$\frac{\partial u}{\partial t} + \underbrace{\frac{\partial L}{\partial u} - \nabla \cdot \Big(\frac{\partial L}{\partial \nabla u} \Big)}_{\text{LHS of EL eqn.}} = 0, \quad u(x,0) = u_0(x)$$

Example: $L = \frac{1}{2}\|\nabla u\|_2^2$

EL eqn: $\Delta u = 0$ } Laplace eqn

Gradient descent PDE: $\frac{\partial u}{\partial t} + \Delta u = 0$ heat eqn

Similarly if $L = \frac{1}{2}\|\nabla u\|^2 - f(x)\,u$

EL eqn: $\Delta u = -f$ (Linear Poisson eqn)

Gradient descent PDE: $\frac{\partial u}{\partial t} - \Delta u = f$

$\therefore$ Heat eqn is gradient descent of Dirichlet Energy w.r.t. L^2 inner product/distance

EL eqⁿ with additional Integral Equality Constraint

$$\min_{u \in C^1(\Omega)} I(u) = \int_\Omega L(\underline{x}, u, \nabla u)\, d\underline{x}$$

s.t. $\displaystyle\int_\Omega \underline{M}(\underline{x}, u, \nabla u)\, d\underline{x} = \underline{K}$ ← element-wise integration

EL eqⁿ: Consider Augmented Lagrangian:

$$L + \langle \underline{\lambda}, \underline{M} \rangle = L + \underline{\lambda}^T \underline{M}$$

$\underline{\lambda}$ is constant vector for integral equality constraint

$\underline{\lambda}$ is called "Lagrange Multiplier"

dimension of $\underline{\lambda}$ is same as number of integral equality constraints.

$\therefore$ EL eq$\underline{\underline{n}}$:

$$\frac{\partial}{\partial u}\left(L + \underline{\lambda}^T \underline{M}\right) - \nabla \cdot \frac{\partial \left(L + \underline{\lambda}^T \underline{M}\right)}{\partial \nabla u} = 0$$

$\underline{\lambda}$: Lagrange Multiplier

L : "Lagrangian"

Example (Dido/Isoperimetric Problem)

maximize $\quad I(u) = \int_{-a}^{+a} u(x)\,dx\,, \qquad 0 < 2a < l$

$u(\cdot) \in C^1(\Omega)$

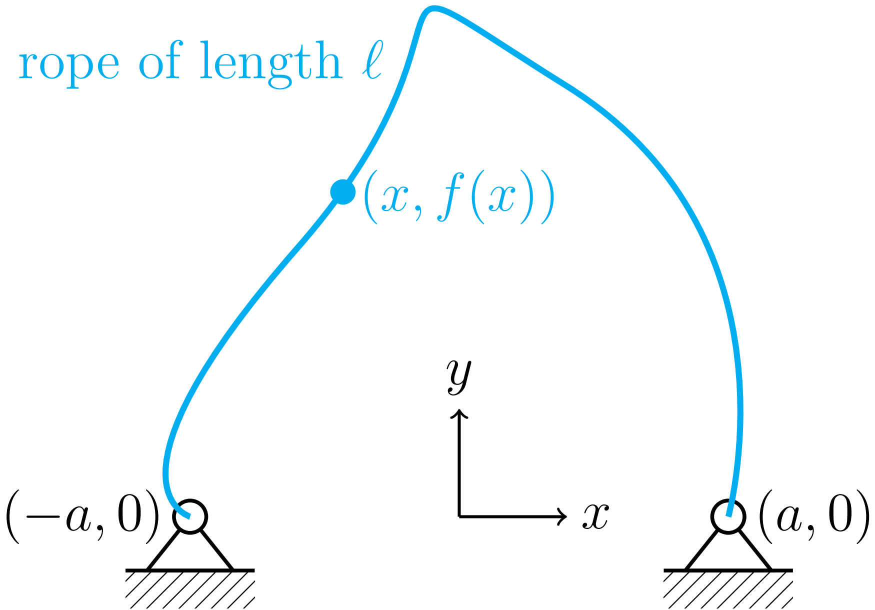
s.t. $\displaystyle\int_{-a}^{+a} \sqrt{1 + (u')^2}\, dx = l$ (given)

B.C. $u(-a) = u(+a) = 0$

$L = -u$

$M = \sqrt{1 + (u')^2}\, dx$

CoV example: Isoperimetric problem



EL eqⁿ:

$$\frac{\partial}{\partial u}(L+\lambda M) - \frac{d}{dx}\frac{\partial}{\partial u'}(L+\lambda M) = 0$$

$$\Rightarrow \quad -1 \quad - \lambda \frac{d}{dx}\left\{ \frac{\cancel{2}u'}{\cancel{2}\sqrt{1+(u')^2}} \right\} = 0$$

$$\Rightarrow \quad 1 + \frac{\lambda \quad u''}{\left[1+(u')^2\right]^{3/2}} = 0$$

Set $u' = \tan\theta \Rightarrow u'' = \sec^2\theta \dfrac{d\theta}{dx}$ (by chain rule)

$\therefore$ EL eqⁿ becomes:

$$1 + \lambda \cos\theta \frac{d\theta}{dx} = 0$$

$$\Rightarrow dx = -\lambda \cos\theta \, d\theta \Rightarrow x = -\lambda \sin\theta + c_1$$

const.

Now use the integral constraint:

$$l = \int_{x=-a}^{x=+a} \sqrt{1+(u')^2}\, dx$$

$$= \int_{\theta = \arcsin(a/\lambda)}^{\theta = -\arcsin(a/\lambda)} \cancel{\sec(\theta)} \left(-\lambda \cancel{\cos}\theta\right) d\theta$$

$$= 2\lambda \arcsin\left(\frac{a}{\lambda}\right)$$

$\therefore$ λ solves transcendental eq^n: $\sin\left(\frac{l}{2\lambda}\right) = \frac{a}{\lambda}$.

On the other hand,

$$du = \tan\theta\, dx = -\lambda \sin\theta\, d\theta$$

$$\Rightarrow y = u(x) = \lambda \cos\theta + \underbrace{c_2}_{\text{const.}}$$

$$\Rightarrow (c_1 - x)^2 + (y - c_2)^2 = \lambda^2 \quad (\text{circular arc})$$

To determine c_1, c_2, use B.C.

$$0 = u(-a) \Leftrightarrow (c_1 + a)^2 + c_2^2 = \lambda^2$$
$$0 = u(+a) \Leftrightarrow (c_1 - a)^2 + c_2^2 = \lambda^2$$

solve for c_1, c_2:

$$c_1 = 0$$
$$c_2 = \sqrt{\lambda^2 - a^2}$$

circular arc shape is optimal

If we have Pointwise equality constraint:
(not integral equality constraint)

Then

$$\min_{u \in C^1(\Omega)} I(u) = \int_\Omega L(\underline{x}, u, \nabla u)\, dx$$

s.t. $\boxed{\underline{M}(\underline{x}, u, \nabla u) = \underline{0} \quad \forall\, \underline{x} \in \Omega}$.

Augmented Lagrangian: (in this case):

$$L + \langle \lambda(\underline{x}), M \rangle = L + \underline{\lambda}^T(\underline{x})\, \underline{M}(\underline{x}, u, \nabla u)$$

Apply EL on this augmented Lagrangian

Here $\underline{\lambda}$ depends on $\underline{x}$

Multi-degree of freedom EL eqⁿ: (doF)

Consider when u is a vector function (i.e.) $u \in \mathbb{R}^n$, but x is scalar (later, "time")

$$L(x, \underline{u}, \underline{u'})$$

derivative of vector w.r.t. scalar

This means $\underline{u}(x)$ is a curve or signal or trajectory in $\mathbb{R}^n$

Multi-doF EL eqⁿ:

$$\underbrace{L_{u_i} = \frac{d}{dx} L_{u'_i}}, \quad i = 1, \ldots, n$$

System of ODEs (n×1 vector ODE)

Newtonian Mechanics & Principle of Least Action

Newton's Law in 3D

$$\boxed{m\ddot{\underline{q}} = \text{Force applied} = \underbrace{-U_q}_{\text{External force}}} \quad (1)$$

6 1st order ODEs

$\underline{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $= -\nabla_{\underline{q}} U(\underline{q})$

$U(\underline{q})$ is some potential that gives some conservative force $= -\nabla_{\underline{q}} U$

EL eqn:

$$L(t, \underbrace{\underline{q}}_{3\times1}, \underline{\dot{q}}_{3\times 1})$$

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Multi dof EL eqn:

$$\boxed{\underbrace{L_{\underline{q}}}_{\frac{\partial L}{\partial \underline{q}}} = \frac{d}{dt} \underbrace{L_{\underline{\dot{q}}}}_{\frac{\partial L}{\partial \underline{\dot{q}}}}} \quad ---(2)$$

(gradient of L w.r.t. $\underline{q}$)

$\underline{q} \in \mathbb{R}^3$