Lecture # 3

Euler-Lagrange (EL) eq$^{\underline{n}}$.

$u : \Omega \mapsto \mathbb{R}$

$\Omega \subset \mathbb{R}^n$

For ($n=1$):

$$\frac{\partial L}{\partial u} - \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) = 0$$

For ($n>1$)

$$\frac{\partial L}{\partial u} - \nabla \cdot \left(\frac{\partial L}{\partial \nabla u}\right) = 0$$

L : "Lagrangian"

$$\min_{u(\cdot) \in C^1(\Omega)} I(u) = \int_\Omega L(\underline{x}, u, \nabla u)\, d\underline{x}$$

integrand

In general, $L \equiv L(\underline{x}, u, \nabla u)$

Spl. case: $n=1$, suppose L does NOT explicitly depend on $\underline{x}$

Then EL eq$^{\underline{n}}$ for $n=1$ could be simplified:

$$\frac{\partial L}{\partial u} = \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right)$$ [EL eq$^{\underline{n}}$ for $n=1$]

$$\Rightarrow \boxed{u' \frac{\partial L}{\partial u} = u' \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right)}$$ [multiplying both sides by u']

On the other hand by chain rule: $\boxed{\frac{dL}{dx} = \frac{\partial L}{\partial u} u' + \frac{\partial L}{\partial u'} u'' + \cancel{\frac{\partial L}{\partial x}}}$ 0

Combining the red & blue box :

$$\frac{dL}{dx} - \frac{\partial L}{\partial u'} u'' = u' \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right)$$

(bring everying in LHS)

$$\Rightarrow \frac{d}{dx}\left(L - u' \frac{\partial L}{\partial u'}\right) = 0$$

$$\Rightarrow \boxed{L - u' \frac{\partial L}{\partial u'} = \text{constant}}$$

(Beltrami identity)

Generalization of EL eq^n for higher order derivatives :

We do for 1D :

EL eq^n (If $L(x,u,u')$) : $\frac{\partial L}{\partial u} = \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right)$

$$\Leftrightarrow L_u = (L_{u'})'$$

EL eq^n (If $L(x, u, u', u'', u''', u'''', \dots, u^{(n)})$)

Then $L_u = (L_{u'})' - (L_{u''})'' + (L_{u'''})''' - \dots = \sum_{r=1}^{n} (-1)^{r+1} \left(L_{u^{(r)}}\right)^{(r)}$

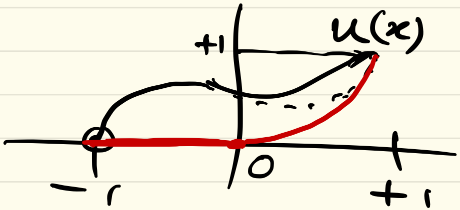
$$\Rightarrow \boxed{L_u = \sum_{r=1}^{n} (-1)^{r+1} \left(L_{u^{(r)}}\right)^{(r)}}$$

The sol$^{\underline{n}}$ of EL eq$^{\underline{n}}$ $u(\cdot)$ may NOT be $C^2(\Omega)$

Example: (1D) [$u''(\cdot)$ may NOT exist]

$$\min_{u(\cdot) \in C^1([-1,+1])} I(u) = \int_{-1}^{+1} \underbrace{u^2 (2x - u')^2}_{L(x,u,u')} dx$$

s.t. $u(-1) = 0$, $u(1) = 1$



Verify that sol$^{\underline{n}}$ of EL eq$^{\underline{n}}$. :

$$u^*(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ x^2 & \text{if } x \in (0, 1] \end{cases}$$

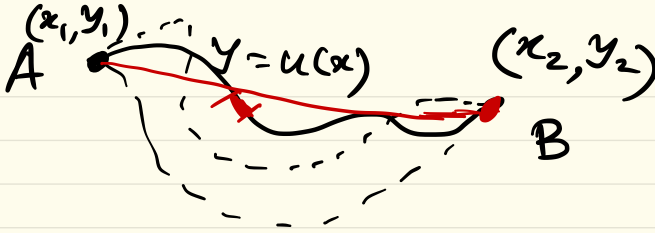
When $u^*(\cdot) \in C^2(\Omega)$

Hilbert's Thm.

If $\frac{\partial^2 L}{\partial u'^2} \neq 0$ in the entire $\mathrm{dom}(u) \equiv \Omega$, $u^*(\cdot) \in C^2(\Omega)$, and then the extremal is called <u>nonsingular</u>.

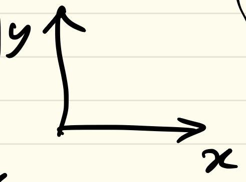
<u>Corollary</u>: Suppose $u^*(\cdot)$ is nonsingular and L is $C^3(\Omega)$ w.r.t. u', then $u^*(\cdot)$ is the <u>unique extremal</u>.

Example: (Shortest Path)



Recall:

$$\boxed{\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}}$$



$$\min_{\substack{u(\cdot) \\ \in C^1}} \int_A^B ds = \int_{x=x_1}^{x=x_2} \underbrace{\sqrt{1 + (u')^2}}_{L(x,u,u')}\, dx$$

$$\boxed{\text{we want to find } y = u(x)}$$

Lagrangian $L = \sqrt{1 + (u')^2}$

Apply EL eqⁿ:

$$\cancelto{0}{\frac{\partial L}{\partial u}} - \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{1 \cdot \cancel{2} u'}{\cancel{2}\sqrt{1 + (u')^2}}\right) = 0 \Rightarrow \boxed{\frac{u''}{\{1 + (u')^2\}^{3/2}} = 0}$$

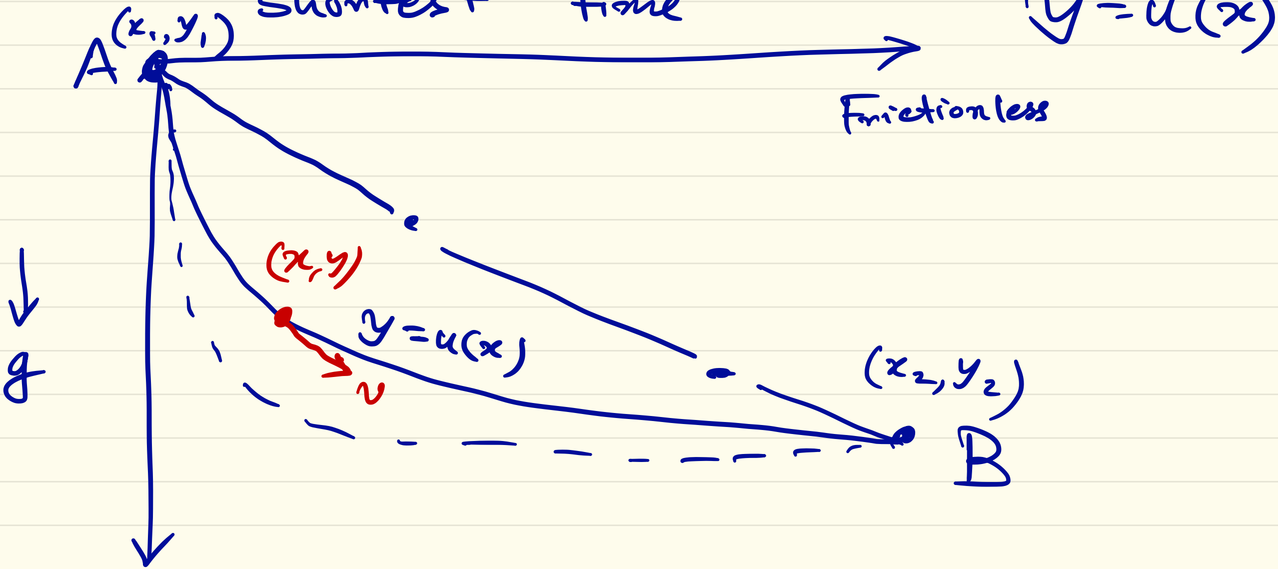
$$\Rightarrow \boxed{u'' = 0}$$

$$\Rightarrow \quad u'' = 0 \quad \Rightarrow \quad u' = c_1 \quad \Rightarrow \quad \underline{u(x) = c_1 x + c_2}$$

$$\left.\begin{aligned} y_1 &= u(x_1) = c_1 x_1 + c_2 \\ y_2 &= u(x_2) = c_1 x_2 + c_2 \end{aligned}\right\} \rightarrow \text{Subtract:} \quad \boxed{c_1 = \frac{y_1 - y_2}{x_1 - x_2}}$$

$$\& \quad c_2 = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}.$$

Example ($\underbrace{\text{Brachisto}}_{\text{shortest}}\underbrace{\text{chrone}}_{\text{time}}$)



$$\min_{u(\cdot)\in C^1(\Omega)} I(u) = \int_A^B dt$$

$$\boxed{v = \frac{ds}{dt}}$$

$$\Rightarrow dt = \frac{ds}{v} = \frac{\sqrt{1+(u')^2}\,dx}{\sqrt{2g}\sqrt{u}}$$

$$\therefore L(x,u,u') = \frac{1}{\sqrt{2g}}\sqrt{\frac{1+(u')^2}{u}}$$

Beltrami identity :

$$L - u'\frac{\partial L}{\partial u'} = \underbrace{c}_{\text{constant}}$$

$$\Rightarrow \sqrt{\frac{1+(u')^2}{u}} - u'\frac{\cancel{2}u'}{\sqrt{u}\cdot\cancel{2}\sqrt{1+(u')^2}} = c$$

$$\Rightarrow \frac{\sqrt{1+(u')^2}}{\sqrt{u}} - \frac{(u')^2}{\sqrt{u}\sqrt{1+(u')^2}} = c$$

$$\Rightarrow \frac{1+\cancel{(u')^2}-\cancel{(u')^2}}{\sqrt{u}\sqrt{1+(u')^2}} = c \Rightarrow \frac{1}{\sqrt{u}\sqrt{1+(u')^2}} = c$$

$$\Rightarrow \sqrt{1+(u')^2} = \frac{1}{c\sqrt{u}}$$

use Energy (E) conservation

$$E = \underbrace{T}_{\text{(Kinetic energy)}} + \underbrace{V}_{\text{(Potential energy)}}$$

$$= \boxed{\frac{1}{2}mv^2 + (-mgy)}$$

$$@A,\ E\big|_{@A} = \frac{1}{2}m\cdot(0)^2 + (-mg\cdot 0) = 0$$

$$\therefore E = E\big|_{@A}$$

$$\Rightarrow \frac{1}{2}\cancel{m}v^2 = \cancel{m}gy \Rightarrow v = \sqrt{2gy} = \sqrt{2gu(x)}$$

$$\Rightarrow (u')^2 = \frac{1}{c^2 u} - 1$$

$$\Rightarrow u' = \sqrt{\frac{1 - c^2 u}{c^2 u}} = \sqrt{\frac{1/c^2 - u}{u}}$$

Let $k := \frac{1}{c^2} = \text{constant}$

$$\Rightarrow u' = \sqrt{\frac{k - u}{u}}$$

1st order nonlinear ODE.

Substitute : $u = k \sin^2 \phi$

$$\Rightarrow \frac{du}{dx} = \sqrt{\frac{k-u}{u}} = \cot(\phi)$$

By chain rule : $\frac{d\phi}{dx} = \frac{d\phi}{du} \frac{du}{dx} = \frac{1}{2k \sin\phi \cos\phi} \cdot \cot\phi$

$$= \frac{1}{2k \sin^2(\phi)}$$

$$\Rightarrow dx = 2k \sin^2 \phi \, d\phi$$

$$\Rightarrow x = k \int (1 - \cos 2\phi) \, d\phi = k\phi - \frac{k}{2}\sin(2\phi) + c_1$$

$$\Rightarrow (x, u(x)) = \left(k\phi - \frac{k}{2}\sin(2\phi) + c_1 , \frac{k}{2}(1 - \cos(2\phi))\right)$$

@ pt. $A(0,0)$: $u = k\sin^2\phi \Rightarrow \phi = 0$

$$\cancelto{0}{x} = k\cancelto{0}{\phi} - \frac{k}{2}\sin(\cancelto{0}{2\phi}) + c_1$$

$$\Rightarrow \boxed{c_1 = 0}$$

Introducing $a := \frac{k}{2}$, and $\theta := 2\phi$,

we get

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y \equiv u(x) &= a(1 - \cos\theta) \end{aligned}\right\} \theta \in (0, \pi).$$

These are parametric eq^{n}s of $\boxed{\text{Cycloid}}$

Exercise: Show that $t^* = \min t = \min \int_A^B dt$

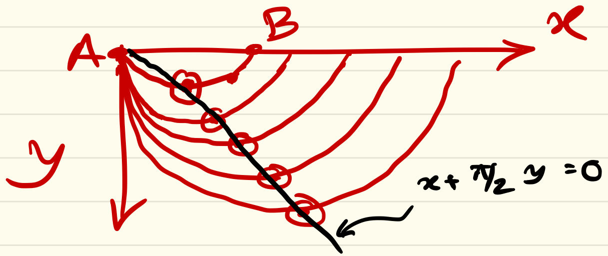
is independent of (x_1, y_1)

$\Leftrightarrow$ Cycloid curve is an isochrone/tautochrone

Exercise: Show that the straight line

$x + \pi/2\, y = 0$ passes through

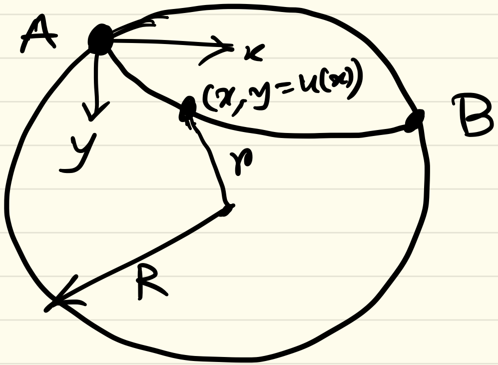
the minima of all Brachistochrone curves



If $x_2 + \frac{\pi}{2} y_2 > 0$, then slide down, then climb

If $x_2 + \pi/2\, y_2 \le 0$, then slide downhill

Brachistochrone with variable g :



2D

uniform interior density

$$E\Big|_{@A} = \frac{1}{2}m\underbrace{v^2}_{0} + V(r)$$

$$= \frac{GMm\,\cancel{r^2}\to R^2}{2R^3}$$

$$= \frac{GMm}{2R}$$

Let $g := \underbrace{\frac{GM}{R^2}}_{\text{accel}^{n} \text{ on surface}}$

$$E\Big|_{@A} = E\Big|_{@(x,u(x))}$$

$$\Rightarrow \frac{GMm}{2R} = \frac{1}{2}mv^2 + \frac{GMm}{2R^3}r$$

$$\Rightarrow v = \sqrt{\frac{GM(R^2-r^2)}{R^3}} = \sqrt{\frac{g}{R}}\sqrt{R^2-x^2-u^2}$$

$$dt = \frac{ds}{v} = \sqrt{\frac{R}{g}} \frac{\sqrt{1+(u')^2}\, dx}{\sqrt{R^2 - x^2 - u^2}}$$

$$\therefore \min_{u(\cdot) \in C^1(A \to B)} \int_A^B dt = \sqrt{R/g} \int_{x_1}^{x_2} \underbrace{\sqrt{\frac{1+(u')^2}{R^2-x^2-u^2}}}_{L(x,u,u')} dx$$

Optimal sol$^{\underline{n}}$ of EL eq$^{\underline{n}}$ $\quad y = u(x)$

$$\begin{cases} x(\theta) = R\left[(1-b)\cos\theta + b\cos\left(\frac{1-b}{b}\theta\right)\right] \\ y(\theta) = R\left[(1-b)\sin\theta - b\sin\left(\frac{1-b}{b}\theta\right)\right] \end{cases}$$

$b \in [0,1]$

Hypocycloid

Example $u : \Omega \mapsto \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ — Dirichlet energy

$$\min_{u \in C^1(\Omega)} I(u) = \int_\Omega \underbrace{\frac{1}{2} \|\nabla u\|_2^2}_{L(\underline{x}, u, \nabla u)} \, d\underline{x}$$

s.t. $u(x) = g(x)$ for $x \in \partial\Omega$.

~~EL eq~~ :

$$\cancel{\frac{\partial L}{\partial u}}^{0} - \nabla \cdot \frac{\partial L}{\partial \nabla u} = 0$$

$$\Rightarrow \quad - \nabla \cdot \nabla u = 0$$

$$\Rightarrow \quad \boxed{\Delta u = 0 \text{ with B.C. } u = g \ @ x \in \partial\Omega}$$

(Laplace eq^n)

Laplacian

$$\Delta = \nabla \cdot \nabla$$

$$= \left(\frac{\partial}{\partial x_1} \ \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n} \right) \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Example:

$$\min_{u \in C^4(\Omega)} I(u) = \int_\Omega \underbrace{\left\{ \frac{1}{2} \|\nabla u\|^2 - f(x)\, u \right\}}_{L(\underline{x},\, u,\, \nabla u)} d\underline{x}$$

$$\frac{\partial L}{\partial u} - \nabla \cdot \frac{\partial L}{\partial \nabla u} = 0$$

$$\Rightarrow -f(x) - \nabla \cdot \nabla u = 0$$

$$\Rightarrow \boxed{\Delta u = -f(x)}$$

(Linear Poisson eq^n)

Example:

$$\min_{u \in C^1(\Omega)} I(u) = \int_\Omega \underbrace{\left\{ \frac{1}{2} \|\nabla u\|^2 - f(u) \right\}}_{L(\underline{x},\, u,\, \nabla u)} d\underline{x}$$

$$\Rightarrow \frac{\partial L}{\partial u} - \nabla \cdot \frac{\partial L}{\partial \nabla u} = 0$$

$$\Rightarrow \Delta u = -\cancel{f'(u)} \rightarrow \phi(u) \Rightarrow \boxed{\Delta u = -\phi(u)}$$

(Nonlinear Poisson eq^n)