Lecture # 2

OPT

$$\min_{\underline{x} \in \mathbb{R}^n} f(\underline{x})$$

Necessary condⁿ for optimality

$$\nabla_{\underline{x}} f(\underline{x}) = 0$$

CoV

$$\min \; I(u) = \int L(\underline{x}, u, \nabla u) \, d\underline{x}$$

$$u(\underline{x}) \in F(\mathbb{R}^n) \qquad \mathrm{dom}(u)$$

$$\subseteq C^1(\mathbb{R}^n)$$

$$\boxed{\begin{gathered} \frac{\partial L}{\partial u} - \nabla \cdot \frac{\partial L}{\partial \nabla u} = 0 \\ u : \mathbb{R}^n \mapsto \mathbb{R} \end{gathered}}$$

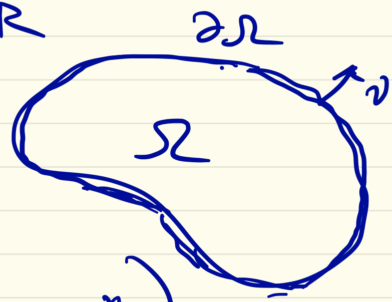
Euler-Lagrange equation

Preparatory Stuff:

$\langle \underline{a}, \underline{b} \rangle = \underline{a}^T \underline{b}$

① Divergence Theorem : (A version)

Suppose $\Omega \subset \mathbb{R}^n$ be open, bounded set with boundary $\partial\Omega$. $u : \Omega \mapsto \mathbb{R}$



If $u \in C^1(\overline{\Omega})$

and F is C^1 vector field

($\text{i.e. } F_i \in C^1(\Omega) \; \forall\, i = 1, \ldots, n$)

Then

$$\boxed{\int_\Omega u(x)(\nabla_{\underline{x}} \cdot F(x))\, d\underline{x} = \int_{\partial\Omega} u(x)\, F(x) \cdot \nu(\underline{x})\, dS - \int_\Omega \langle \nabla u(x), F(x) \rangle\, d\underline{x}}$$

Vanishing Lemma: Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $a(\underline{x}) \in C(\Omega)$

If $$\int_\Omega a(\underline{x})\, b(\underline{x})\, d\underline{x} = 0 \quad \text{for all } b(\underline{x}) \in C^\infty(\Omega)$$

then $a(\underline{x}) = 0$ for all $\underline{x} \in \Omega$.

Proof: Suppose to the contrary that $\exists\, \underline{x}_0 \in \Omega$ s.t. $a(\underline{x}_0) \neq 0$.

Assume, w.l.o.g., that $\epsilon := a(\underline{x}_0) > 0$.

Since $a(\underline{x})$ is continuous, there exists $\delta > 0$, s.t. $a(\underline{x}) \geq \frac{\epsilon}{2}$ whenever $\| \underline{x} - \underline{x}_0 \| < \delta$

Now, let $b(\underline{x}) \in C^{\infty}(\Omega)$ be a test $f^{\underline{n}}$ satisfying

$$b(\underline{x}) \begin{cases} > 0 & \text{for } \|\underline{x} - \underline{x}_0\| < \delta \\ = 0 & \text{for } \|\underline{x} - \underline{x}_0\| \geq \delta \end{cases}$$

Then,

$$\underline{0} = \int_{\Omega} a(\underline{x})\, b(\underline{x})\, d\underline{x} = \int_{B(\underline{x}_0, \delta)} a(\underline{x})\, b(\underline{x})\, d\underline{x} \geq \frac{\epsilon}{2} \int_{B(\underline{x}_0, \delta)} b(\underline{x})\, d\underline{x}$$

$$\underline{> 0}$$

contradiction.

Remark : Test $f^{\underline{n}}$ $b(\underline{x})$ satisfying the proof construction is

$$b(\underline{x}) = \begin{cases} \exp\left(- \dfrac{1}{\delta^2 - \|\underline{x} - \underline{x}_0\|_2^2} \right) & \text{for } \|\underline{x} - \underline{x}_0\| < \delta \\ 0 & \text{otherwise} \end{cases}$$

Actual Result

Statement (Euler - Lagrange (EL) equation)

Let $\Omega := \mathrm{dom}(u(\underline{x})) \subset \mathbb{R}^n$.

Consider the problem

$$\min_{u(\cdot) \in C^1(\Omega)} I(u),$$

where $I(u) = \int_{\Omega} L(\underline{x}, u, \nabla u)\, d\underline{x}$

s.t. Boundary Condition (B.C.) : $u(\underline{x}) = \overline{u}(\underline{x})$ for $\underline{x} \in \partial\Omega$.

(e.g. $y_1 = u(x_1)$, $y_2 = u(x_2)$)

Then,

the necessary condition for $I(u)$ to attain minimum is

$$\boxed{\frac{\partial L}{\partial u} - \nabla \cdot \frac{\partial L}{\partial \nabla u} = 0}$$

Proof: $L : \underset{\subset \mathbb{R}^n}{\overset{\underset{\sim}{x}}{\Omega}} \times \mathbb{R} \times \mathbb{R}^n \longmapsto \mathbb{R}$

(We call L as "Lagrangian".

- Suppose minimum of $I(u) = \int_{\Omega} L(\underset{\sim}{x}, u, \nabla u)\, d\underset{\sim}{x}$ is attained at $u^*(\underset{\sim}{x})$ that respects the B.C. $u^*(\underset{\sim}{x}) = \bar{u}(\underset{\sim}{x})$ for all $\underset{\sim}{x} \in \partial\Omega$

  (e.g. $u(x_1) = y_1$, $u(x_2) = y_2$)

- Consider deforming or a variation of $u^*(\underset{\sim}{x})$ while maintaining feasibility (i.e. satisfying B.C.)

- Let us ONLY consider deformation of a particular type, described by a parameterized family of functions

$$\boxed{u(\epsilon, \underline{x}) := u^*(\underline{x}) + \epsilon \phi(\underline{x})}$$

where ϵ in $\mathbb{R}$ (real number)

$\phi(\underline{x}) \in C^\infty(\Omega)$ is arbitrary but fixed once the arbitrary choice is made

For $u(\epsilon, x)$ to be feasible, we must have

(If B.C. $\underline{y}_1 = u(\underline{x}_1)$, $\underline{y}_2 = u(\underline{x}_2)$)

$$u(\epsilon, \underline{x}_1) = \underline{y}_1$$

$$\Rightarrow \cancel{u^*(\underline{x}_1)}^{\,y_1} + \epsilon \phi(\underline{x}_1) = \underline{y}_1$$

$$\Rightarrow \phi(\underline{x}_1) = 0, \quad \phi(\underline{x}_2) = 0.$$

In general, $\phi(\underline{x}) = 0$ for all $\underline{x} \in \partial\Omega$.

- Then $I(u) \equiv I(u^* + \epsilon\phi) \equiv I(\epsilon)$

$$= \int_{\Omega} L\left(\underline{x}, u(\epsilon, \underline{x}), \nabla_{\underline{x}} u(\epsilon, \underline{x})\right)$$

- Clearly, $I(u^*) \leq I(\epsilon)$ for all ϵ in $\mathbb{R}$

- We know, for optimality, we need that

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(\epsilon) = 0 \Leftrightarrow \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(u^* + \epsilon\phi) = 0$$

(*) RHS for 1D ($n=1$) looks like

$$\frac{\partial L}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial L}{\partial u}\frac{\partial u}{\partial \epsilon} + \frac{\partial L}{\partial u'}\frac{\partial u'}{\partial \epsilon}$$

• Now,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(u^* + \epsilon\phi)$$

$$= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_\Omega L\Big(\underline{x}, \underbrace{u^* + \epsilon\phi}_{u}, \underbrace{\nabla_{\underline{x}}(u^* + \epsilon\phi)}_{\nabla u}\Big)\, d\underline{x}$$

$$= \int_\Omega \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L\Big(\underline{x}, \underbrace{u^* + \epsilon\phi}_{u}, \underbrace{\nabla_{\underline{x}}(u^* + \epsilon\phi)}_{\nabla u}\Big)\, d\underline{x}$$

($\because$ limits of integration does NOT depend on ϵ)

By chain rule,

$$\frac{d}{d\epsilon}L(\underline{x}, u, \nabla u) = \left\langle \nabla_{\underline{x}} L, \cancelto{0}{\frac{\partial \underline{x}}{\partial \epsilon}} \right\rangle + \left\langle \frac{\partial L}{\partial u}, \cancelto{\phi}{\frac{\partial u}{\partial \epsilon}} \right\rangle + \left\langle \frac{\partial L}{\partial \nabla u}, \cancelto{\nabla\phi}{\frac{\partial \nabla u}{\partial \epsilon}} \right\rangle$$

(*)

Therefore,

$$\frac{d}{d\epsilon} L(\underline{x}, u, \nabla u) = \frac{\partial L}{\partial u}\phi + \left\langle \frac{\partial L}{\partial \nabla u}, \nabla \phi \right\rangle$$

$$\therefore \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(\epsilon) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(u^* + \epsilon\phi)$$

$$= \int_\Omega \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(\underline{x}, u, \nabla u)\, d\underline{x}$$

$$= \underbrace{\int_\Omega \frac{\partial L}{\partial u}\phi\, dx}_{1^{st}\text{ term}} + \underbrace{\int_\Omega \left\langle \frac{\partial L}{\partial \nabla u}, \nabla\phi \right\rangle dx}_{2^{nd}\text{ term}}$$

Recall that

$$\phi(\underline{x}) = 0 \text{ for all } \underline{x} \in \partial\Omega.$$

$\therefore$ By Divergence Thm on the 2^{nd} term:

$$\int_\Omega \left\langle \frac{\partial L}{\partial \nabla u}, \nabla \phi \right\rangle d\underline{x} = -\int_\Omega \left(\nabla \cdot \frac{\partial L}{\partial \nabla u}\right) \phi \, d\underline{x}$$

$$\therefore \quad 0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} I(\epsilon) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} I(u^* + \epsilon\phi)$$

$$= \int_\Omega \underbrace{\left(\frac{\partial L}{\partial u} - \nabla \cdot \frac{\partial L}{\partial \nabla u}\right)}\phi \, d\underline{x}$$

$\therefore$ By vanishing Lemma:

$$\boxed{\frac{\partial L}{\partial u} - \nabla \cdot \frac{\partial L}{\partial \nabla u} = 0} \longleftrightarrow \text{EL eq}^n.$$

Proved $\blacksquare$

Remarks :

① Solutions of EL eqn are called critical pt-s (could be multiple; some of them may be maxima, some minima, some saddle)

② For $n=1$, we get ODE :
$$\frac{\partial L}{\partial u} - \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) = 0.$$
with B.C. $u(x_1)=y_1$, $u(x_2)=y_2$

③ For $n>1$, we get PDE :
$$\frac{\partial L}{\partial u} - \nabla\cdot\left(\frac{\partial L}{\partial \nabla u}\right) = 0.$$
with B.C. $u(\underline{x}) = \bar{u}(\underline{x})$ (given) on $x \in \partial\Omega$

Convexity is sufficient for minimum.

$$L(\underline{x}, u, \nabla u) \equiv L(\underline{x}, u, \underline{p})$$

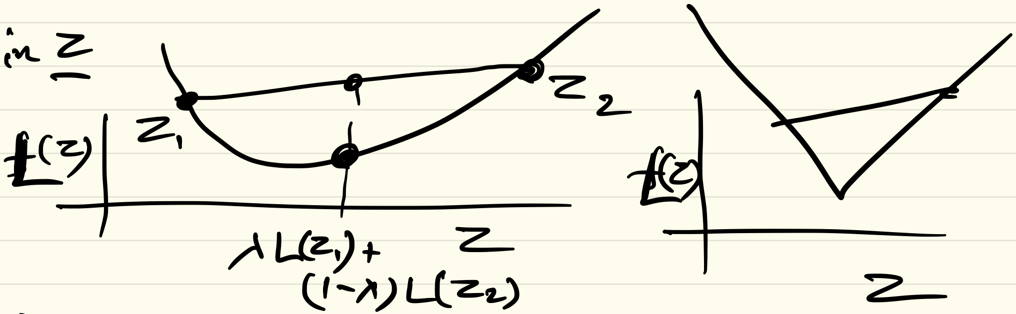
If the mapping $(u, \underline{p}) \mapsto L(\underline{x}, u, \underline{p})$ is jointly convex (i.e. in $\mathbb{R}^{n+1}$) in $(u, \underline{p})$ for each $\underline{x}$

the sol^{n} of EL is a minimum for $I(u)$.

(existence of minimum)

Jointly strictly convex $\Longrightarrow$ unique minimum.

By convex in $\underline{z}$



$$\lambda L(z_1) + (1-\lambda) L(z_2)$$

$$\geqslant L(\lambda z_1 + (1-\lambda) z_2) \text{ for all}$$

$0 \leq \lambda \leq 1$

$z_1, z_2 \in \mathbb{R}^n$

$(>)$ strictly convex.

Example:

- $L(\underline{x}, u, \underline{p}) = \frac{1}{2}\, \underline{p}^T \underline{p} - u\, g(\underline{x})$

  is jointly convex in $(u, \underline{p})$

- $L(\underline{x}, u, \underline{p}) = u\, \underbrace{p_1}_{\text{first component of } \underline{p}}$ (individually convex but not jointly)

- $L(\underline{x}, u, \underbrace{p}_{\text{scalar}}{}^{u'}) = \underbrace{u^2}_{\text{convex in } u} + \underbrace{(p^2 - 1)^2}_{\text{non-convex in } \underline{p}}$

say, $n = 1$

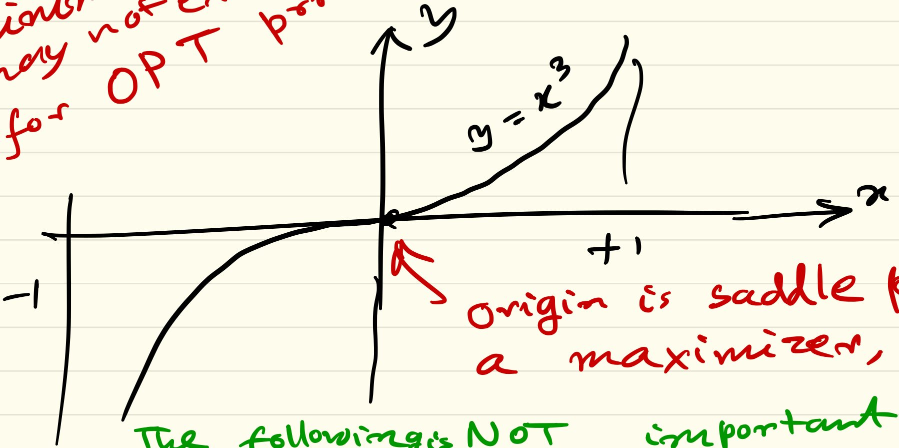
$$\underbrace{\min_{u}\; I(u) = \int_0^1 \left\{ (u(x))^2 + \left((u'(x))^2 - 1 \right)^2 \right\} dx \quad \text{s.t. } u(0) = u(1) = 0}_{\text{CoV Problem}}$$

(Try drawing $p \mapsto (p^2 - 1)^2$)

(In fact, for this CoV problem, minimum does NOT exist)

OPT : $\min\limits_{x \in [-1,1]} x^3 \rightarrow \frac{d}{dx} x^3 = 0 \Rightarrow x^* = 0$

This is not surprising, minimum may not exist for OPT problems too.



origin is saddle point, neither a maximizer, nor a minimizer.

The following is NOT important (can be skipped in first reading)

Aside: Proof of the claim that the CoV problem

$$\min_{u(\cdot)} I(u) = \int_0^1 \left\{ u^2 + \left((u')^2 - 1 \right)^2 \right\} dx, \quad \text{s.t. } u(0) = u(1) = 0$$

has no solution.

see Next page :

Proof : Let $u_k(x)$ be a sawtooth wave of period $2/k$ and amplitude $1/k$.

(i.e.) $u_k(x) = x$ for $x \in [0, 1/k]$

$= 2/k - x$ for $x \in [1/k, 2/k]$

$= x - 2/k$ for $x \in [2/k, 3/k]$

etc.

$u_k(x)$ satisfies $u_k'(x) = 1$ or $u_k'(x) = -1$ at all but finite number of pt.s of non-differentiability

Also, $0 \leq u_k(x) \leq 1/k$

$$\therefore \quad I(u_k) \leq \int_0^1 \frac{1}{k^2}\, dx = \frac{1}{k^2}$$

If the minimum exists, then

$$0 \leq \min_u I(u) \leq 1/k^2 \text{ for all } k = 1, 2, \ldots$$

$$\therefore \quad \min_u I(u) = 0$$

However, such a minimizer needs to satisfy $(u(x))^2 = 0$ and $(u'(x))^2 = 1$ $\forall\, x$. There is no such f^n. $\blacksquare$