Lecture #18

State Estimation for Markov Chain

Markov chain

Transition Probability Matrix : $P = [P_{ij}]$, $i \in \mathcal{X}$

$$p(y|x) \text{ for } y \in \mathcal{Y}$$

Given $p_0(\underline{x}_0) \leftarrow$ initial distribution

Notation :

$$p_{0|-1} := p_0$$

Question : How to go from $p_{k+1|k}$ to $p_{k+1|k+1}$

(Measurement update / Posterior computation)

Can show by Baye's rule :

$$p(x(k+1)=i \mid y^{k+1}) = \frac{p(x(k+1)=i \mid y^{k})\, p(y(k+1) \mid x(k+1)=i)}{\sum_{j \in \mathcal{X}} p(x(k+1)=i \mid y^{k})\, p(y(k+1) \mid x(k+1)=j)} \in [0,1]$$

⓪ How to go from $p_{k|k}$ to $p_{k+1|k}$?

(Time update/ Prior computation)

$$\boxed{\text{We know state } x \sim \text{Markov}(P)}$$

$$p(x(k+1)=i \mid y^k)$$

$$= \sum_{j \in \mathcal{X}} p(x(k)=j \mid y^k)\, p_{ji}$$

(in vector form $\underbrace{p_{k+1|k}}_{1\times n} = \underbrace{p_{k|k}}_{1\times n} P_{n\times n}$)

⓪ Recursive procedure:

$$\underbrace{0|-1}_{\text{Initial distribution}} \longrightarrow \underbrace{0|0 \longrightarrow 1|0}_{\text{Time update}} \underbrace{\longrightarrow 1|1}_{\text{Measurement update}} \xrightarrow[\text{Time update}]{} 2|1 \longrightarrow 2|2 \longrightarrow \cdots$$

⊙ Summary:

$$p_{k+1|k+1}(y(k+1)) = T_k\Big(\underbrace{p_{k|k}(y^k)}_{\text{prev. probability distribution}}, \underbrace{y(k+1)}_{\text{new measurement/ observation}}\Big)$$

$\Leftrightarrow$ We can think of

$p_{k|k}$ = "Hyperstate" of the system /
"Information state" /
"Belief state"

① We can write these updates in vector form:

$$p_{k|k}(y^k) = [p_{k|k}(1|y^k), p_{k|k}(2|y^k), \ldots, p_{k|k}(n|y^k)]$$

$$P = [P_{ij}]$$

$$\underbrace{D(y)}_{\text{Diag. matrix}} = \begin{bmatrix} p(y|1) & & & 0 \\ & p(y|2) & & \\ 0 & & \ddots & \\ & & & p(y|N) \end{bmatrix}$$

Also, $\underline{e} := \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} = \underbrace{\text{ones}(n, 1)}_{\text{MATLAB}}$

Then

$$\underbrace{p_{k+1|k+1}(y^{k+1})}_{\text{measurement update}} = \frac{p_{k+1|k}(y^k)\, D(y(k+1))}{p_{k+1|k}(y^k)\, D(y(k+1))\, e} \Bigg\} \text{Nonlinear due to normalization}$$

Time update:

$$\underbrace{p_{k+1|k}(y^k)}_{1\times n} = \underbrace{p_{k|k}(y^k)}_{1\times n}\ \underbrace{P}_{n\times n}$$

These 2 equations define an algorithm ("filter") for the "state estimation" problem

filtering eq$^{\underline{n}}$s

Estimation Problem with Controlled Markov chain:

$$[P_{ij}(u)] = P(u)$$

$$u(k) = \gamma_k(y^k, u^{k-1})$$

This argument is not necessary to write if policy $\gamma_k(\cdot)$ is deterministic

Define:

$$\underbrace{Z^k := (y^k, u^{k-1})}_{\text{Past information}}$$

Now we have:

$$\underbrace{p_{k+1|k+1}(Z^{k+1})}_{\text{only depends on action, NOT on policy}} \propto p_{k|k}(Z^k)\, P(u(k))\, \underline{\underline{D(y(k+1))}}$$

up to normalization

① Optimal Control with Partial Observation:

$P(u) = [P_{ij}(u)]$, $p_{0|-1}$: initial distribution

$p(y|x)$

$\gamma \in \Gamma \equiv$ History dependent policies

History of past observation:

$$u(k) = \gamma_k(y^k, u(k-1))$$

$$= \gamma_k(Z^k)$$

Cost:

$$\min_{\gamma(\cdot) \in \Gamma} \mathbb{E}\Big[\underbrace{c_N(x_N)}_{\text{terminal cost}} + \underbrace{\sum_{k=0}^{N-1} c_k(x(k), u(k))}_{\text{running cost}}\Big]$$

want to find $\gamma^*(\cdot)$

Result: Optimal policy is a "separated/separation policy".

$$u^*(k) = \gamma_k^*\big(\underbrace{p_{k|k}(Z^k)}_{\text{current belief}}\big)$$

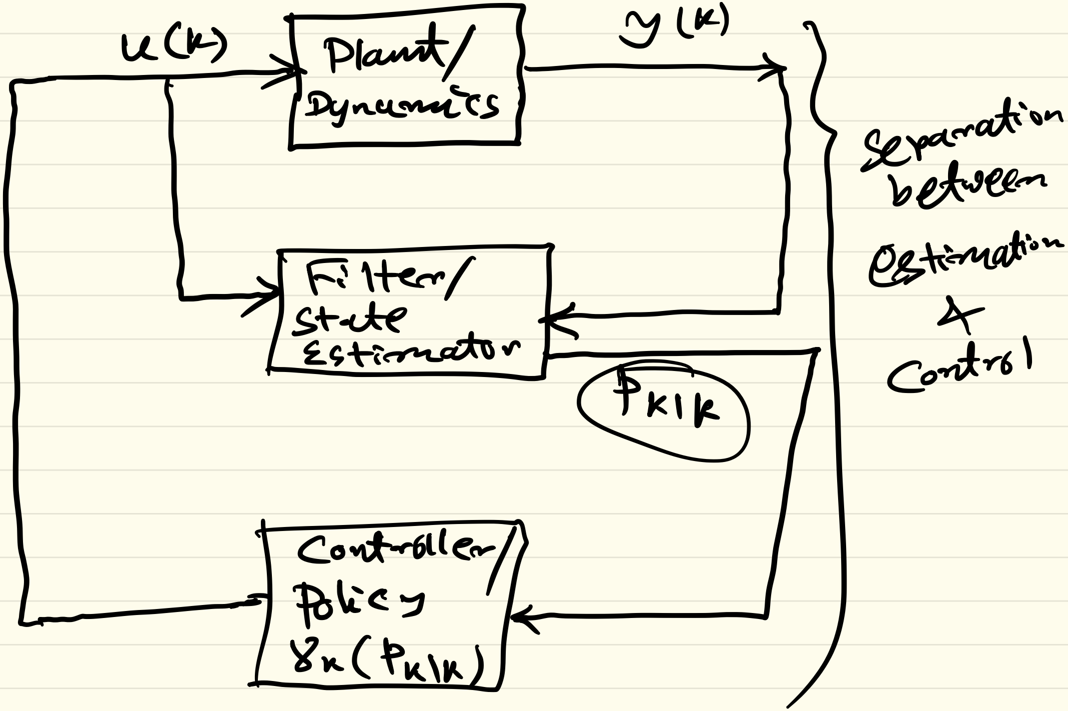
Much simpler than

$$u(k) = \gamma_k(Z^k)$$

The result says that a <u>separated policy</u> is optimal in the class of all history dependent policies.

(separated $\equiv$ separation of estimation/filtering & control)

Therefore, conceptually, we can do the following:



① Define: (Expected cost-to-go under policy γ).

$$J_k^{\gamma} := \mathbb{E}^{\gamma}\left[c_N(x(N)) + \sum_{l=k}^{N-1} c_l(x(l), u(l)) \,\middle|\, Z^k \right]$$

History of past observation action pairs

Theorem: (DP eq^ns / Backward Recursion)

Define the following:

$$V_N(\pi) := \sum_{i \in \mathcal{X}} \pi(i)\, c_N(i)$$

π: Probability distribution

$$V_k(\pi) := \min_{u \in \mathcal{U}} \Big[\sum_{i \in \mathcal{X}} \pi(i)\, c_k(i,u) + \underbrace{\sum_{i \in \mathcal{X}} \pi(i) \sum_{j \in \mathcal{X}} p_{ij}(u) \sum_{y \in \mathcal{Y}} p(y|j)\, V_{k+1}\left(T_{k+1}(\pi, y, u)\right)}_{\mathbb{E}[V_{k+1}(\pi_{k+1})]} \Big]$$

Then,

(1) $J_k^{\gamma} \geq V_k(p_{k|k}) \quad \forall\ \gamma \in \Gamma$

$\Leftrightarrow$ (Answer of DP recursion is optimal)

(2)

Suppose

$\gamma_k^*(\pi)$ attains the minimum for each π.

Then the separated policy $\gamma_k^*(p_{k|k})$

is optimal.

Proof: Clearly, this is true for $k = N$.

(Strategy: Backward induction) $\underbrace{\text{Suppose this is true for } (k+1)}_{\text{inductive hypothesis}}$.

Now, we're going to show that it's true for k.

P.T.O.

$$J_k^{\gamma} = \mathbb{E}^{\gamma}\Big[c_N(x(N)) + \sum_{\ell=k}^{N-1} c_\ell(x(\ell),u(\ell)) \,\Big|\, Z^k\Big]$$

$$= \mathbb{E}^{\gamma}\Big[\Big\{c_k(x(k),u(k)) + \mathbb{E}^{\gamma}\Big[\sum_{\ell=k+1}^{N-1} c_\ell(x(\ell),u(\ell)) + c_N(x(N)) \,\Big|\, Z^{k+1}\Big]\Big\} \,\Big|\, Z^k\Big]$$

$$\geq \mathbb{E}^{\gamma}\Big[c_k(x(k),u(k)) + V_{k+1}\big(p_{k+1|k+1}(Z^{k+1})\big) \,\Big|\, Z^k\Big]$$

$$= \mathbb{E}^{\gamma}\Big\{\mathbb{E}^{\gamma}\Big[c_k(x(k),u(k)) + V_{k+1}\big(T_k(p_{k|k}(Z^k), y(k+1), u(k))\big) \,\Big|\, Z^k, u(k)\Big] \,\Big|\, Z^k\Big\}$$

$$= \mathbb{E}^{\gamma}\Big[\sum_{i\in\mathcal{X}} p_{k|k}(i|Z^k)\, c_k(i,u(k)) + \sum_{i\in\mathcal{X}} p_{k|k}(i|Z^k) \sum_{j\in\mathcal{X}} P_{ij}(u(k)) \sum_{y\in\mathcal{Y}} p(y|i)\, V_{k+1}(p_{k|k}(Z^k), y, u) \,\Big|\, Z^k\Big]$$

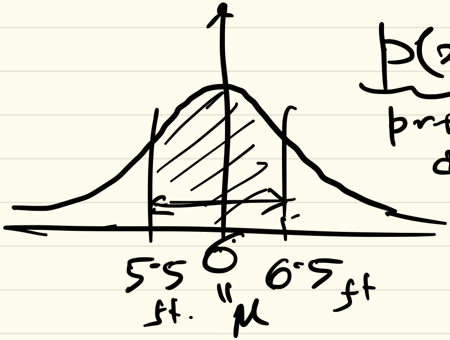
P.T.O.

$$\geq \min_{u \in \mathcal{U}} \left\{ \sum_{i \in \mathcal{X}} p_{k|k}(i \mid Z^k)\, c_k(i,u) + \sum_{i \in \mathcal{X}} p_{k|k}(i, Z^k) \sum_{j \in \mathcal{X}} p(j \mid i)\, V_{k+1}\big((p_{k|k} \mid Z^k), j, u\big) \right\}$$

$\blacksquare$

LQG $\Leftrightarrow$ Linear Quadratic Gaussian Control

Gaussian or Normal Distribution



$$\underbrace{p(x)}_{\text{probability density function.}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\Updownarrow$$

$$x \sim \underbrace{\mathcal{N}}_{\text{Normal distribution}}(0,1)$$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \Leftrightarrow x \sim \mathcal{N}(\mu, \sigma^2)$$

1D.

Multivariate Normal:

$$p(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

$\boxed{\underline{x} \in \mathbb{R}^n}$

$$\Updownarrow$$

$$\underline{x} \sim \mathcal{N}(\underline{\mu}, \Sigma)$$

Facts: $\underline{x} \sim \mathcal{N}(\underline{\mu}, \Sigma)$

(i) $\mathbb{E}[\underline{x}] = \underline{\mu}$ or, $\overline{x}$

(ii) $\mathrm{cov}(\underline{x}) = \mathbb{E}[(\underline{x}-\underline{\mu})(\underline{x}-\underline{\mu})^T] =: \Sigma$

covariance

(iii) $\mathbb{E}[\underline{x}\,\underline{x}^T] = \Sigma + \underline{\mu}\,\underline{\mu}^T$

(iv) $\mathbb{E}[A\underline{x}] = A\underline{\mu}$,

(v) If $\underline{x} \sim \mathcal{N}(\underline{\mu}, \Sigma)$ then $A\underline{x} \sim \mathcal{N}(A\underline{\mu}, A\Sigma A^T)$

(vi) $\mathbb{E}[\underline{x}^T Q \underline{x}]$ where $\underline{x} \sim \mathcal{N}(\underline{\mu}, \Sigma)$

$$= \mathbb{E}[\mathrm{tr}(\underline{x}^T Q \underline{x})]$$

$$= \mathbb{E}[\mathrm{tr}(\underline{x}\underline{x}^T Q)]$$

$$= \mathrm{tr}\ \mathbb{E}[\underline{x}\underline{x}^T Q]$$

$$= \mathrm{tr}((\mathbb{E}[\underline{x}\underline{x}^T]) Q)$$

$$= \mathrm{tr}((\underline{\mu}\underline{\mu}^T + \Sigma) Q)$$

$$= \boxed{\mathrm{tr}(\underline{\mu}^T Q \underline{\mu}) + \mathrm{tr}(\Sigma Q)}$$

(vii) Suppose 2 vectors: $\underline{x}$ & $\underline{z}$ are jointly Gaussian/Normal:

$$\begin{Bmatrix} \underline{x}_{n\times 1} \\ \underline{z}_{n\times 1} \end{Bmatrix} \sim \mathcal{N}\left(\underbrace{\begin{Bmatrix} \underline{\mu}_x \\ \underline{\mu}_z \end{Bmatrix}}_{2n\times 1}, \underbrace{\begin{bmatrix} \Sigma_{xx} & \Sigma_{xz} \\ \Sigma_{zx} & \Sigma_{zz} \end{bmatrix}}_{2n \times 2n} \right)$$

We say

$\underline{x}$ & $\underline{z}$ are uncorrelated $\Rightarrow \Sigma_{xz} = 0$

or $\Sigma_{zx} = 0$.

- Lemma:

(if jointly Gaussian)

uncorrelated $\Longrightarrow$ Independent

(always true)

- We want to do MMSE (Minimum Mean Square Error)

Estimation :

Suppose $\underline{x}$ & $\underline{y}$ have some joint distribution (not necessarily Gaussian)

→ We observe "y". (noisy)

→ Based on $\underline{y}$, we want to make an estimate of $\underline{x}$

→ Call that estimate $\underline{g}(\underline{y})$

→ We want optimal estimate (in MMSE sense)

→ (i.e.) We want $\underline{g}^*(\underline{y})$ so that

$$\boxed{\min_{g(\cdot) \in G} \| \underline{x} - g(y) \|^2 \quad G \leftarrow \text{all possible functions.}}$$

Claim: $g^*(\underline{y}) = \underbrace{\mathbb{E}[\underline{x} \mid \underline{y}]}$ (conditional mean)

This is a random vector.

Proof: Take any $\underline{g}(\underline{y})$.

Then

$$\mathbb{E}\left[\|\underline{x} - g(\underline{y})\|^2\right]$$

$$= \mathbb{E}\left[\|\underline{x} - \underline{g}^*(\underline{y}) + \underline{g}^*(\underline{y}) - \underline{g}(\underline{y})\|^2\right]$$

$$= \mathbb{E}\|\underline{x} - g^*(\underline{y})\|^2 + \mathbb{E}\|g^*(\underline{y}) - g(\underline{y})\|^2 +$$

Now expand

$$2\,\mathbb{E}\left[\underbrace{(x - g^*(y))^T (g^*(y) - g(y))}\right]$$

want to show: $\longrightarrow$ This is $= 0$.

Now,

$$\mathbb{E}\left[\mathbb{E}\left[(\underline{x}-\underline{g}^*(\underline{y}))^T(\underline{g}^*(\underline{y})-\underline{g}(\underline{y}))\mid\underline{y}\right]\right]$$

$$=\mathbb{E}\Big[\underbrace{\mathbb{E}\left[(\underline{x}-g^*(y))^T\mid y\right]}_{0}\left(g^*(\underline{y})-\underline{g}(\underline{y})\right)\Big]$$

$$=0. \quad \blacksquare$$

⊙ Estimation when $\underline{x}$ & $\underline{y}$ are jointly Gaussian

Suppose $\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}\underline{\mu}_x\\ \underline{\mu}_y\end{pmatrix}, \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy}\\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}\right)$

What is $\mathbb{E}[\underline{x}\mid\underline{y}]$?

Define:

$$\underline{\hat{x}} := \underline{\mu}_x + \Sigma_{xy}\Sigma_{yy}^{-1}(\underline{y} - \mu_y)$$

Also $\underline{\tilde{x}}$:= Error $= \underline{x} - \underline{\hat{x}}$

$$= \underline{x} - \underline{\mu}_x - \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)$$

But $\underline{\hat{x}}$ is a linear transformation of $\underline{y}$

$\underline{\tilde{x}}$ " " " " " $\underline{y}$.

$\therefore$ all these are Gaussian

Now we prove that $\Sigma_{\tilde{x}y} = 0$

(Then we invoke that $\underline{\tilde{x}}$ & $\underline{y}$ are indep.)

$$\mathbb{E}[\tilde{x} \mid y] = \mathbb{E}[\tilde{x}] = 0.$$

$$\mathbb{E}\left[x - \underline{\mu}_x - \Sigma_{xy}\Sigma_{yy}^{-1}(\mu - \mu_y) \,\middle|\, y\right] = 0$$

$$\mathbb{E}\left[\underline{x} \,|\, \underline{y}\right] = \mathbb{E}\left[\underline{\mu}_x + \Sigma_{xy}\Sigma_{yy}^{-1}(\underline{\mu} - \underline{\mu}_y) \,\middle|\, \underline{y}\right]$$

$$= \underline{\mu}_x + \Sigma_{xy}\Sigma_{yy}^{-1}(\underline{\mu} - \underline{\mu}_y)$$

Then

$$\Sigma_{\underline{\tilde{x}}\,\underline{y}} = \mathbb{E}\left[(\underline{\tilde{x}} - 0)(\underline{y} - \underline{\mu}_y)^T\right]$$

$$= \Sigma_{xy} - \Sigma_{xy}\cancel{\Sigma_{yy}}^{-1}\cancel{\Sigma_{yy}}$$

$$= 0$$

$$\boxed{\therefore\ \mathbb{E}\left[\underline{x} \,|\, \underline{y}\right] = \underline{\mu}_x + \Sigma_{xy}\Sigma_{yy}^{-1}(\underline{y} - \underline{\mu}_y)}$$

Also, Error $\underset{\sim}{\tilde{x}}$ & $\underline{y}$ are indep.

(i.e.) $\mathbb{E}[\underline{\tilde{x}}\,\underline{\tilde{x}}^T \mid \underline{y}] = \mathbb{E}[\underline{\tilde{x}}\,\underline{\tilde{x}}^T]$

(saying, in Gaussian case, there is no such thing as "more informative data/observation" or "less informative observation")

Also, $\mathbb{E}[\underline{\tilde{x}}\,\underline{\tilde{x}}^T]$

$$= \Sigma_{xx} + \Sigma_{xy}\Sigma_{yy}^{-1}\cancel{\Sigma_{yy}}\,\cancel{\Sigma_{yy}^{-1}}\Sigma_{yx} - 2\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}$$

$$= \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}.$$

Similarly, we can do $\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$ jointly Gaussian.