Lecture # 14

Regarding singular controls:

Main message: Singular controls happen often in practice.

Example (1)

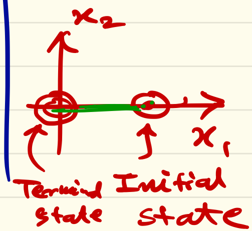
Singular controls in Nonlinear Systems

$$\min_{u(\cdot)} \int_0^T 1 \, dt$$

s.t. $\dot{x}_1 = x_2^2 - 1$

$\dot{x}_2 = u$

$|u| \leq 1$



$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ given

$\begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ fixed.

$H = 1 + \lambda_1(x_2^2 - 1) + \lambda_2 u$

$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \Rightarrow \lambda_1 = \text{const}.$

$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -2\lambda_1 x_2$

PMP

$$u^* = \begin{cases} -1 & \text{if } \lambda_2(t) > 0 \\ +1 & \text{if } \lambda_2(t) < 0 \\ ?? & \text{if } \lambda_2(t) = 0 \end{cases}$$

Can show:

$u^*(t) = 0 \;\; \forall t \in [0, T]$

$\therefore$ Optimal control is singular for all $t \in [0, T]$

$\Leftrightarrow$ $x^*(t)$ is a singular arc for all $t \in [0, T]$

Slightly general set up for singular optimal control in nonlinear systems:

$$\boxed{\underline{x} \in \mathbb{R}^n, \quad u \in \mathbb{R}.} \qquad \dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})u$$

$$\min_{u(\cdot)} \int_0^T 1 \, dt$$

$$\text{s.t.} \quad \dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})u$$

$$|u| \leq 1.$$

$$H = 1 + \underline{\lambda}^T(\underline{f}(\underline{x}) + \underline{g}(\underline{x})u)$$

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} = -\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^T \underline{\lambda} - \left(\frac{\partial \underline{g}}{\partial \underline{x}}\right)^T \underline{\lambda}\, u$$

PMP

$$u^* = \begin{cases} -1 & \text{if } \langle \underline{\lambda}(t), \underline{g}(\underline{x}) \rangle > 0 \\ +1 & \text{if } \langle \underline{\lambda}(t), \underline{g}(\underline{x}) \rangle < 0 \\ ?? & \text{if } \langle \underline{\lambda}(t), \underline{g}(\underline{x}) \rangle = 0 \end{cases}$$

singular

∴ Whether $u^*(t)$ is singular or not, depends on "switching f^n"

$$\sigma(t) := \langle \underbrace{\underline{\lambda}(t)}_{n\times 1}, \underbrace{\underline{g}(\underline{x})}_{n\times 1} \rangle$$

If the control $u^*(t)$ is singular over some subinterval $[t_1, t_2]$ then

$$\sigma(t) := \langle \underline{\lambda}(t), \underline{g}(\underline{x}) \rangle \equiv 0 \quad \forall\, t \in [t_1, t_2]$$

$$\Rightarrow \underline{\lambda}(t) \perp^{r} \underline{g}(\underline{x})$$

Also,

$$\dot{\sigma}(t) \equiv 0 \quad \forall\, t \in [t_1, t_2]$$

$$\ddot{\sigma}(t) \equiv 0 \quad \forall\, t \in [t_1, t_2]$$

$$\vdots$$

etc.

Now, $\dot{\sigma}(t) := \dfrac{d}{dt}\sigma(t) = \langle \dot{\underline{\lambda}}, \underline{g} \rangle + \langle \underline{\lambda}, \dot{\underline{g}} \rangle$

$$= \Big\langle \underbrace{-\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial \underline{x}} u\right)^T \underline{\lambda}}_{\text{costate ODE RHS}}, \underline{g} \Big\rangle + \Big\langle \underline{\lambda}, \underbrace{\frac{\partial g}{\partial \underline{x}} \dot{\underline{x}}}_{\text{chain rule}} \Big\rangle$$

$$\left(\underline{f}(\underline{x}) + \underline{g}(\underline{x})u\right)$$

$$\Rightarrow \dot{\sigma} = \frac{d}{dt}\sigma(t) = \left\langle \underline{\lambda}, -\left(\frac{\partial \underline{f}}{\partial \underline{x}} + \frac{\partial \underline{g}}{\partial \underline{x}} u\right) \underline{g} \right\rangle + \left\langle \underline{\lambda}, \frac{\partial \underline{g}}{\partial \underline{x}} \left(\underline{f}(\underline{x}) + \underline{g}(\underline{x}) u\right) \right\rangle$$

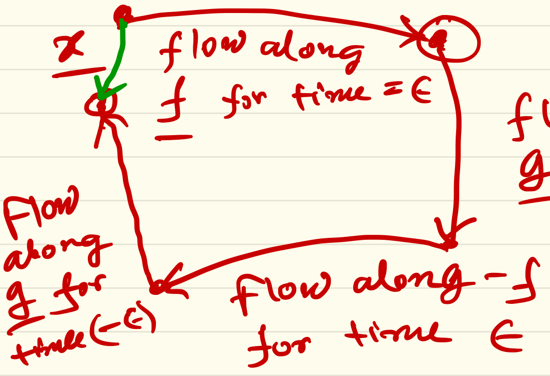
$$= \left\langle \underline{\lambda}, -\frac{\partial \underline{f}}{\partial \underline{x}} \underline{g} + \frac{\partial \underline{g}}{\partial \underline{x}} \underline{f} \right\rangle \equiv 0 \quad \forall\, t \in [t_1, t_2]$$

$$= \left\langle \underline{\lambda}, \underbrace{[\underline{f}, \underline{g}](\underline{x})}_{\substack{\text{new vector field} \\ \text{called "Lie Bracket"} \\ \text{or "commutator" vector field} \\ \text{of } f \,\&\, \underline{g}}} \right\rangle$$

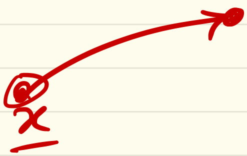
Obviously,

$$[\underline{f}, \underline{g}](\underline{x}) = -[\underline{g}, \underline{f}](\underline{x}), \quad [f, f] = 0.$$

What does it mean: $[\underline{f}, \underline{g}](\underline{x})$



$h(\underline{x})$



Can show that this composition: $\epsilon^4 [\underline{f}, \underline{g}](\underline{x})$

Example: $\underline{f}(\underline{x}) = A\underline{x}, \quad \underline{g}(\underline{x}) = B\underline{x}$

Then
$$[\underline{f}, \underline{g}](\underline{x}) = \frac{\partial \underline{g}}{\partial \underline{x}} \underline{f}(\underline{x}) - \frac{\partial \underline{f}}{\partial \underline{x}} \underline{g}(\underline{x})$$
$$= B(A\underline{x}) - A(B\underline{x})$$
$$= \underbrace{(BA - AB)\underline{x}}$$
$$= [\underline{f}, \underline{g}](\underline{x})$$

Again, having singular control u^* is equivalent to

$$\sigma(t) \equiv 0 \ \forall t \in [t_1, t_2] \Leftrightarrow \langle \underline{\lambda}(t), \underline{g}(\underline{x}) \rangle \equiv 0,$$

$$\dot{\sigma}(t) \equiv 0 \ \forall t \in [t_1, t_2] \Leftrightarrow \langle \underline{\lambda}(t), [\underline{f}, \underline{g}](\underline{x}) \rangle \equiv 0,$$

$$\ddot{\sigma}(t) \equiv 0 \ \forall t \in [t_1, t_2] \Leftrightarrow \langle \underline{\lambda}(t), [\underline{f}, [\underline{f}, \underline{g}]](\underline{x}) \rangle + \langle \underline{\lambda}(t), [\underline{g}, [\underline{f}, \underline{g}]](\underline{x}) \rangle u^* \equiv 0,$$

$\vdots$

$\vdots$

<u>Theorem for $n=2$</u>: Let min-time problem, $|u| \leq 1$, $n=2$. Suppose $g(\underline{x})$ & $[f, g](\underline{x})$ are linearly independent (for all $\underline{x} \in \mathbb{R}^n$), (i.e.) $\text{rank}[\underline{g} \,|\, [\underline{f}, \underline{g}]] = 2$.

then u^* is bang-bang.. (no singular control).

Back to example 1:

$$\left.\begin{aligned}\dot{x}_1 &= x_2^2 - 1\\ \dot{x}_2 &= u\end{aligned}\right\} \Leftrightarrow \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \underbrace{\begin{pmatrix}x_2^2-1\\ 0\end{pmatrix}}_{\underline{f}(\underline{x})} + \underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}_{\underline{g}(\underline{x})} u$$

$$\underline{x} :\equiv \begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \mathbb{R}^2, \quad u \in \mathbb{R}.$$

$$\underline{g}(\underline{x}) = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

$$[\underline{f}, \underline{g}](\underline{x}) := \frac{\partial \underline{g}}{\partial \underline{x}} \underline{f}(\underline{x}) - \frac{\partial \underline{f}}{\partial \underline{x}} \underline{g}(\underline{x})$$

$$= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \underline{f}(\underline{x}) - \begin{pmatrix}0 & 2x_2\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}0\\ 0\end{pmatrix} - \begin{pmatrix}2x_2\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}-2x_2\\ 0\end{pmatrix}$$

$$\text{rank}\left[\underline{f} \mid [\underline{f}, \underline{g}]\right](\underline{x}) = \text{rank}\begin{bmatrix}0 & -2x_2\\ 1 & 0\end{bmatrix}$$

If $x_2 = 0$, then NOT linearly indep. $\Leftarrow \boxed{u^* = 0}$

Example (Fuller's Problem)

$$\min_{u(\cdot)} \int_0^T x_1^2 \, dt, \quad T \text{ free.}$$

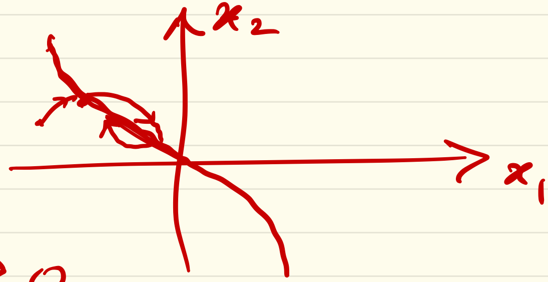
$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}\right\} \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \text{ given}, \quad \begin{pmatrix} x_1(T) \\ y_1(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{fixed.}$$

Exercise : show :

1. No singular arc.
2. $u^* \in \{-1, 1\}$, i.e., bang-bang with # of switching $= +\infty$.
3. Switching takes place on the curve $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 + \gamma |x_2| x_2 = 0 \right\}$ where $\gamma \simeq 0.445$
4. Time intervals betn consecutive switches decreses in Geometric Progression.

$\Rightarrow$ Fuller's phenomenon, also known as Zeno behavior

In the <u>min. time problem</u>, switching curve had the same form with $\gamma = 1/2$.



We can encode this type of problems as:

$$\min_{u(\cdot)} J(u) = \int_0^T |x_1(t)|^\nu \, dt, \quad \nu \geq 0$$

If $\begin{cases} \nu = 0, & \text{then min. time OCP, \# of switching} \leq 1 \\ \nu = 2, & \text{Fuller's problem, \# of switching} = +\infty \end{cases}$

What happens in between:

<u>Bifurcation</u>: $\exists\ \bar{\nu} \approx 0.35$ s.t. for $\nu \in [0, \bar{\nu}]$ u^* is bang-bang with ≤ 1 switch

If $\nu > \nu^* \approx 0.35$, then zeno behavior (∞ switching)

Maximum Principle

Theory: PMP, costates, inequalities, 2PBVP

Setting: Deterministic
BOTH continuous & discrete time

Computation: 2 ways

- Indirect Method
  - Exact OCP
  - Approx. sol^n of problem → Shooting type algorithms
- Direct Method (After 1990's)
  - Approximate the problem
  - Solve Approx. Problem exactly

Exact 2PBVP

Dynamic Programming

Theory: Recursion on Value Function

Setting: Deterministic and Stochastic case
(BOTH continuous & discrete time)

continuous time: PDE
discrete time: Recursion on function space

Computation: Exponential complexity