Lecture #13

Bang-bang for general LTI.

$$\min_{u(\cdot)} \int_0^T 1 \, dt$$

s.t. $\dot{x} = Ax + Bu$, $\underline{x} \in \mathbb{R}^n$, $\underline{u} \in \mathbb{R}^m$.

$\underline{x}(0) = \underline{x}_0$ given.
$\underline{x}(1) = \underline{x}_1$ given.

$$\boxed{|u_i| \leq 1 \;\; \forall i = 1, \ldots, m \;\; \Updownarrow \;\; \underline{u} \in [-1, +1]^m}$$

$$H = 1 + \underline{\lambda}^T (A\underline{x} + \underline{\underline{B}}\,\underline{u})$$

$$\boxed{\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} = -A^T \underline{\lambda}}$$

PMP

$$\underline{\lambda}^T B \underline{u} = \sum_{i=1}^{m} \langle \underline{\lambda}(t), \underline{b}_i \rangle u_i$$

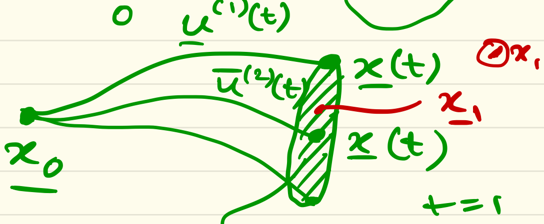
columns of B matrix

necessary condition for feasibility

Assumption :

$$\boxed{\underline{x}_1 \in \overrightarrow{\text{Reach}}(\underline{x}_0)}$$

$$x(t) = \exp(At)\,\underline{x}_0 + \int_0^t e^{A(t-\tau)} B\, \underline{u}(\tau)\, d\tau$$



$\overrightarrow{\text{Reach}}(\underline{x}_0)$
(Forward reach set)

(Less restrictive condition than controllability)

Since each u_i can be chosen indep.ly:

$$\therefore \quad u_i^*(t) = -\operatorname{sgn}\left(\langle \underline{\lambda}(t), \underline{b}_i \rangle\right)$$

$$= \begin{cases} +1 & \text{if } \langle \underline{\lambda}(t), \underline{b}_i \rangle < 0 \\ -1 & \text{if } \langle \underline{\lambda}(t), \underline{b}_i \rangle > 0 \\ ?? & \text{if } \langle \underline{\lambda}(t), \underline{b}_i \rangle = 0 \end{cases}$$

(indeterminate)

"singular control"

(corresponding part of state trajectory is called a "singular arc")

If "singular control" can be ruled out, we say the OCP is "normal" (otherwise the OCP is "abnormal")

From costate ODE:

$$\underline{\lambda}(t) = \exp\left(A^T(T-t)\right)\lambda(T)$$

$$\Rightarrow \underline{\langle \underline{\lambda}(t), \underline{b}_i \rangle} = \underbrace{\langle \underline{\lambda}(T), \exp(A(T-t))\underline{b}_i \rangle,}_{\text{Real analytic function of } t,}$$

(Take inner product with $\underline{b}_i$, $i=1,\dots,m$)

$\forall\, i = 1, \dots, m$

$\therefore$ can only have finitely many zeros in any subinterval of $[0,T]$

$\Rightarrow$ Finitely many switching

(If a real analytic f^n is zero then all its derivatives are also identically zero).

The following result was proved by Pontryagin et. al. (1962)

- An LTI system is normal ($\Leftrightarrow$ no singular control) if

$$\text{rank}\left[\underline{b}_i \mid A\underline{b}_i \mid \cdots \mid A^{n-1}\underline{b}_i\right] = n$$

$$\forall\, i = 1, \ldots, m$$

where $\underline{b}_i :=$ i^{th} column of B matrix.

$\Leftrightarrow$ $(A, \underline{b}_i)$ is controllable for all $i = 1, \ldots, m$

(More restrictive than (A, B) controllable).

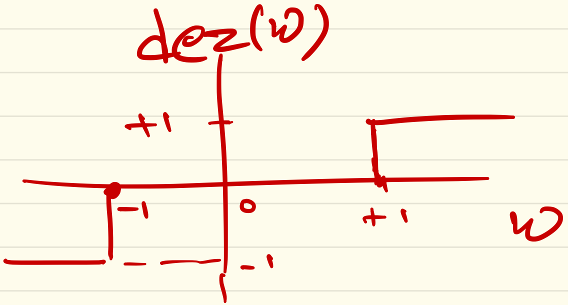
Corollary: If (A, B) is normal ($\Leftrightarrow (A, \underline{b}_i)$ controllable $\forall\, i = 1, \dots, m$) then $u^*(t)$ is bang-bang.

Bang-off-Bang

Dead-zone-function

$$\mathrm{dez}(w) = \begin{cases} -1 & \text{if } w < -1 \\ -1 < \quad < 0 & \text{if } w = -1 \\ 0 & \text{if } -1 < w < +1 \\ 0 < \quad < +1 & \text{if } w = +1 \\ +1 & \text{if } w > +1 \end{cases}$$

singular control



If $w(t) = \pm 1$ over finite time sub-interval, then abnormal solⁿ $\Leftrightarrow$ singular control

Example: Double integrator (with Bang-Off-Bang)

$$\ddot{x} = u \iff \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \quad \Bigg| \quad \min_{u(\cdot)} \int_0^T |u(t)|\, dt \quad \Bigg| \quad |u| \leq +1$$

$(x_1(0), x_2(0)) = (x_0, y_0)$ (given)

$(x_1(T), x_2(T)) = (0,0)$ fixed

T is fixed

(If T is free, then in general u^* may NOT exist)

$$\underline{\psi} = \begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$H = |u| + \underline{\lambda_1 x_2} + \lambda_2(t) u$$

$$\left.\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \end{aligned}\right\} \quad \begin{aligned} &\lambda_1 = \text{const.} \\ &\lambda_2(t) = \lambda_2(T) + (T - t)\lambda_1 \end{aligned}$$

linear in "t"

PMP: For H to be minimum:

$$u^*(t) = \begin{cases} \boxed{+1 \quad \text{if} \quad \lambda_2(t) < -1} \\ 0 \le \quad \le 1 \quad \text{if} \quad \lambda_2(t) = -1 \\ \boxed{0 \quad \text{if} \quad -1 < \lambda_2(t) < +1} \\ -1 \le \quad \le 0 \quad \text{if} \quad \lambda_2(t) = +1 \\ \boxed{-1 \quad \text{if} \quad \lambda_2(t) > 1} \end{cases}$$

Singular controls

Since $\lambda_2(t)$ is a linear f^n of t,

$\therefore$ $u^* = \pm 1$ cannot switch to $u^* = \mp 1$ without passing through the intermediate value $u^* = 0$.

$\Longleftrightarrow$ at most 2 switching.

Recall,

$$\underset{\sim\sim\sim}{T_{min}} = y_0 + \sqrt{4x_0 + 2y_0^2} \quad \Big\} \quad \underline{\text{check this}}$$

min. time
double integrator
bang - bang

Suppose now, the fixed T in our problem

$$\min_{u(\cdot)} \int_0^T |u|\, dt, \quad T \text{ fixed}, \text{ satisfies}$$

$$\boxed{T > T_{min} = y_0 + \sqrt{4x_0 + 2y_0^2}}$$

(We will see that $T > T_{min}$ is a necessary condition for existence of u^* in Bang-Off-Bang problem for double integrator)

Since $\lambda_2^*(t)$ is linear in t, therefore the optimal control $u^*(t) \in \{-1, 0, +1\}$ given by

$$u^* = \begin{cases} -1 & \text{if} \quad 0 \le t < t_1 \\ 0 & \text{if} \quad t_1 \le t < t_2 \\ +1 & \text{if} \quad t_2 \le t \le T \end{cases}$$

where the switching times t_1, t_2 are to be determined.

Recall that terminal state $\underline{x}(T) = \begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

When $u^* = -1$, then

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -1 \end{aligned}\right\} \Longleftrightarrow \begin{aligned} x_1(t_1) &= x_0 + y_0 t_1 - t_1^2/2 \\ x_2(t_1) &= y_0 - t_1 \end{aligned}$$

integrate

Similarly, $u^* = 0$, then . . . .

" $u^* = +1$, then . . . .

Finally impose terminal constraint :

Show that: (eliminate t_2)

$$t_1^2 - (y_0 + T)t_1 + \left(x_0 + y_0 T + \frac{y_0^2}{2}\right) = 0$$

This gives:

$$t_1 = \frac{(y_0 + T) \pm \sqrt{(y_0+T)^2 - 4\left(x_0 + y_0 T + \frac{y_0^2}{2}\right)}}{2}$$

$$= \frac{(y_0 + T) \pm \sqrt{(y_0 - T)^2 - (4x_0 + 2y_0^2)}}{2}$$

Since $t_1 < t_2$, this means:

t_1 = RHS with $(-)$ sign

t_2 = RHS with $(+)$ sign.

Solⁿ is admissible if stuff under $\sqrt{\cdots} > 0$

$T > T_{min}$

#7 Inequality on f^{n} of state & control:

$$\boxed{C(\underline{x}, \underline{u}, t) \leq 0}$$

$$H = L + \underline{\lambda}^T \underline{f} + \mu C$$

$$\mu = \begin{cases} > 0 & \text{for } C = 0 \\ = 0 & \text{for } C < 0 \end{cases}$$

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} = \begin{cases} -\frac{\partial L}{\partial \underline{x}} - \lambda^T \frac{\partial f}{\partial x} - \mu \frac{\partial C}{\partial x}, & C = 0 \\ -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x}, & C < 0 \end{cases}$$

PMP: $0 = \frac{\partial H}{\partial \underline{u}} = L_u + \underline{\lambda}^T \underline{f}_u + \mu C_u = 0.$

If $C < 0$, then $\mu = 0$, & PMP determines $u^*(t)$.

If $C = 0$, use PMP together with the constraint itself $C = 0$, to solve for $u(t)$ & μ.

⑧ <u>Pure State inequality constraints</u>: [This is from Bryson & Ho's Book]

$$S(\underline{x}, t) \le 0 \quad \text{- - - - } (*)$$

(for simplicity, assume both S and u as scalars)

The idea is to take successive time derivatives of S (perhaps upto "q^{th} order) until "u" appears explicitly. If indeed "q" time-derivatives are required, then we say that (*) is a <u>q^{th} order state inequality constraint</u>.

Now define the Hamiltonian as:

$$H = L + \underline{\lambda}^T \underline{f} + \mu S^{(q)}$$

where $S^{(q)} = 0$ on the constraint boundary $(S=0)$

$\mu = 0$ off the " " $(S<0)$

Necessary condition for $\mu(t)$ is:

$$\mu(t) \geq 0 \text{ on } S=0$$

Since control of $S(x,t)$ is obtained by changing its q^{th} time derivative, no finite control will keep the system on the constraint boundary if the path entering the constraint boundary does not meet the following "tangency" constraints:

$$N(\underline{x},t) := \begin{pmatrix} S(\underline{x},t) \\ S^{(1)}(\underline{x},t) \\ \vdots \\ S^{(q-1)}(\underline{x},t) \end{pmatrix} = \underline{0}_{q\times 1} \quad \text{----} (**)$$

These same tangency constraints apply to the path leaving the constraint boundary.

The eqns $(**)$ form a set of interior boundary conditions as in #5. Consequently, the costates $\underline{\lambda}(t)$ are, in general, discontinuous at junction points between constrained and unconstrained arcs.

One can set the $\underline{\lambda}$'s and H as discontinuous at the entry point $t = t_1$, and continuous at the exit point, W.l.o.g.

The "entry" & "exit" points may or may not be "corners", i.e., places where the control vector is discontinuous.

Example

Bryson - Denham Problem

(URL sent)

Most practical state inequality constrained problems are solved numerically, via Direct OCP solvers such as ICLOCS.