Lecture #12

(Lec. 11 $\leftrightarrow$ Midterm)

#5 Interior Point Constraints:

$$\underbrace{\underline{N}}_{q\times 1}(\underline{x}(t_1), t_1) = 0, \qquad \underbrace{t_0}_{\stackrel{=}{0}} < t_1 < T$$

(vector f^{n}.) (say)

This means we have 3PBVP instead of 2PBVP.

Consider times t_1^- and t_1^+

Then we have 2 extra conditions:

$$\underline{\lambda}(t_1^+) = \frac{\partial L}{\partial \underline{x}(t_1)}$$

and $$H(t_1^+) = -\frac{\partial L}{\partial t_1}$$

These combined with the transversality gives the so-called "Jump conditions" (next page)

$$\boxed{\begin{aligned} \underline{\lambda}(t_1^-) &= \underline{\lambda}(t_1^+) + \pi^T \frac{\partial \underline{N}}{\partial \underline{x}(t_1)} \\ H(t_1^-) &= H(t_1^+) - \pi^T \frac{\partial \underline{N}}{\partial t_1} \end{aligned} \Bigg\} \text{ "Jump conditions"}}$$

where $\underline{\pi}_{q\times 1}$ (column vector of same dimension as $\underline{N}$)

Lagrange multiplier to-be-determined so that the interior point constraint

$$\underline{N}_{q\times 1}(\underline{x}(t_1), t_1) = 0 \text{ is satisfied.}$$

$\boxed{\text{constant vector}}$

"Jump" implies discontinuities in $\underline{\lambda}$ & H @ $t = t_1$

However, the state vector is continuous, i.e.,

$$\underline{x}(t_1^-) = \underline{x}(t_1^+)$$

Can be extended to multiple times:

$t_0 < t_1 < t_2 < T$

t_1, t_2 given (fixed)

$$N(\underline{x}(t_1), t_1) = 0$$
$$M(\underline{x}(t_2), t_2) = 0$$

} Double jump conditions etc.

4PBVP. etc.

Exercise (for #5)

Min. time intercept passing through an intermediate point:

Given (x_1, y_1), x_f is given, a is constant (known), $0 < t_1 < T$, but t_1 is free otherwise

$$\min_{\theta(t)} \int_0^T 1 \cdot dt$$

s.t.

$\dot{x} = u$	$x(0) = 0$	$x(t_1) = x_1$	$x(T) = x_f$
$\dot{u} = a\cos\theta$	$y(0) = 0$	$y(t_1) = y_1$	$y(T) = 0$
$\dot{y} = v$	$u(0) = 0$		
$\dot{v} = a\sin\theta$	$v(0) = 0$		

#6 Control inequality constraints:

$$C(u,t) \le 0$$

$$H = L + \underline{\lambda}^T \underline{f} + \mu^T C$$

PMP: $0 = \dfrac{\partial H}{\partial \underline{u}} = L_u + \underline{\lambda}^T \dfrac{\partial \underline{f}}{\partial \underline{u}} + \mu^T C_u$

additional requirement:

$$\left.\begin{array}{lll} \mu \geq 0 & \text{if} & C = 0 \\ \mu = 0 & \text{if} & C < 0 \end{array}\right\}$$

Bang-Bang Control:

(Linear OCP with linear control inequality constraint)

The dynamics & Lagrangian are both linear in $\underline{u}$

$$\min_{u(\cdot)} \int_0^T L(\underline{x}, \underline{u}, t)\, dt \quad \Big\} \ \text{L is linear in } \underline{u}$$

$$\text{s.t.} \quad \dot{x} = Ax + Bu$$

$$\& \quad u_{min} \le u \le u_{max}$$

$$\Updownarrow$$

$$|u| \le \cancel{u_{max}} \ 1$$

$$\Updownarrow$$

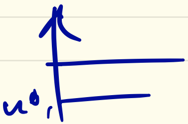
$$\boxed{|u| \le 1}$$

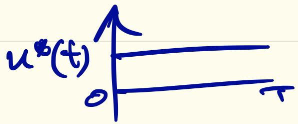
$$H = L + \underline{\lambda}^T \underline{f}$$

also linear in $\underline{u}$ — $A\underline{x} + B\underline{u}$

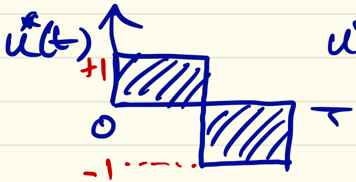
Without loss of generality, we consider

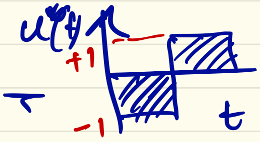
$$\left.\begin{array}{l} u_{min} = -1 \\ u_{max} = +1 \end{array}\right\}$$





Optimal control $u^*(t)$





PMP (Pontryagin Maximum Principle)

$$0 = \frac{\partial H}{\partial \underline{u}} \iff \text{choose } \underline{u} \text{ such that it pointwise } \underline{\text{minimizes}}\ H.$$

- In general, no minimizer $u^*(\cdot)$ exists (since $u^* = -\infty$ will make $H = -\infty$) unless you specify inequality constrains on the state and/or control variables.

- If $|u| \leq u_{max}$, then $u^*(\cdot) \in \{-u_{max}, +u_{max}\}$.

Example: Minimum time control of double integrator, with given (x_0, y_0), and given $(x(T), y(T)) = (0,0)$

i.e., Bring a point mass to origin in minimum time

Problem:

$$\min_{u(\cdot)\in C([0,T])} \int_0^T 1\, dt, \quad T \text{ free.}$$

By $C([0,T])$ we mean either left/right continuity @ jump

(càdlag) functions

$$\ddot{x} = u \iff \begin{array}{l} x_1 := x \\ x_2 := \dot{x} \end{array} \iff \boxed{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{array}}$$

$(x_0, y_0) \equiv (x_1(0), x_2(0))$ given

$\begin{pmatrix} x(T) \\ y(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, fixed.

$$\boxed{|u| \le 1 \iff -1 \le u \le +1}$$

$$H = L + \lambda^T f = 1 + \lambda_1 x_2 + \lambda_2(t) u$$

Costate eqn:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \iff \boxed{\lambda_1 = c_1} \text{ (constant)}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 = -c_1$$

$$\iff \boxed{\lambda_2(t) = -c_1 t + c_2}$$

PMP:

$$u^*(t) = \begin{cases} \overset{u_{min}}{-1} & \text{if } \lambda_2(t) > 0 \\ \overset{u_{max}}{+1} & \text{if } \lambda_2(t) < 0 \\ \text{undetermined (arbitrary)} & \text{if } \lambda_2(t) = 0 \end{cases}$$

$$u^*(t) = -\text{sign}(\lambda_2(t))$$

(any $u(t)$ s.t. $-1 \leq u(t) \leq +1$ is possible)

We will soon see that this case will NOT be possible for our dynamics $\ddot{x} = u$.

Transversality: $d\underline{x}(T) = 0$, $dT \neq 0$.

final state fixed.

$$H(T) = 0 \iff 1 + c_1 \underbrace{\cancel{x_2(T)}}_{0} + \lambda_2(T)\, u(T) = 0$$

$$\iff \boxed{\lambda_2(T)\, u(T) = -1}$$

$\iff$ either $u(T) = 1$ and $\lambda_2(T) = -1$

or $u(T) = -1$ and $\lambda_2(T) = +1$

Notice that $\lambda_2(t)$ depends on initial condition (x_0, y_0)

$\therefore$ Optimal control:

$u^*(t) =$ either $-1 \quad \forall\, t \in [0,T]$

or -1 switching to $+1$ @ $t=t_s$

or $+1$ " " -1 @ $t=t_s$

or $+1 \quad \forall\, t \in [0,T]$

This is because

$\lambda_2(t)$ is a linear function of t

(i.e.) $\lambda_2(t)$ can change sign at most once

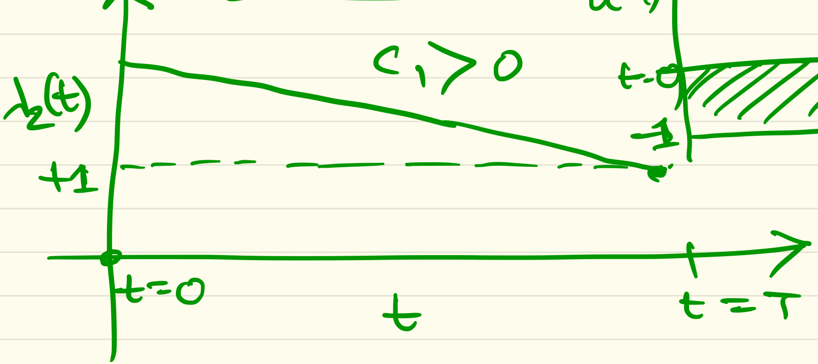
$\therefore u^*(t) = -\text{sign}(\lambda_2(t))$ can also

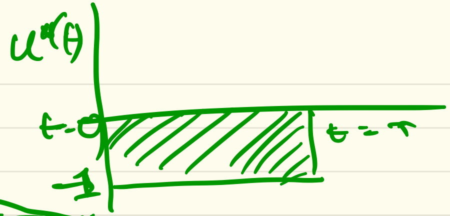
change sign at most once.

$\therefore @ \lambda_2(t) = 0$, $u^*(t) \in \{-1, +1\}$ ($\because u(\cdot) \in C[0,T]$)

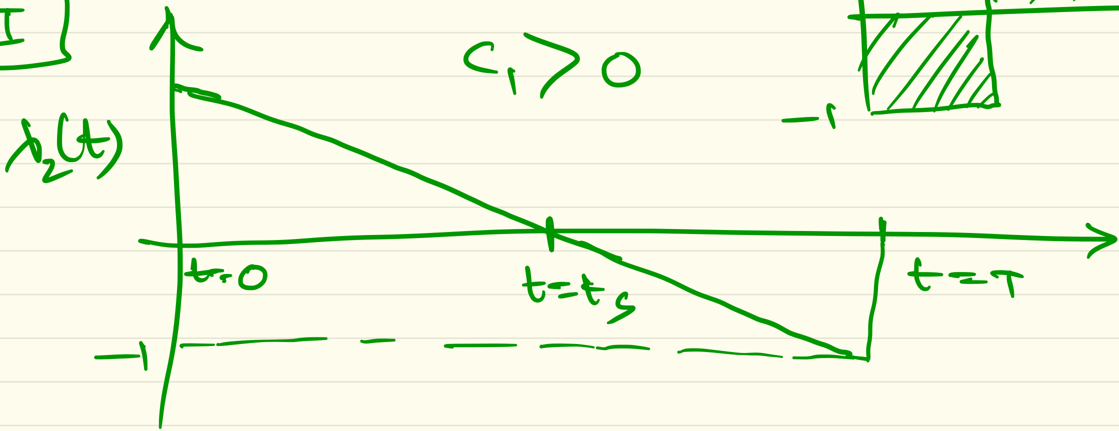
$\therefore u^*(t) \in \{-1, +1\} \quad \forall\, t \in [0,T]$

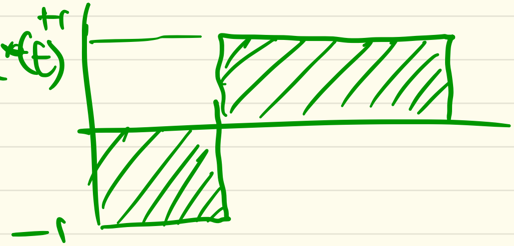
Case I





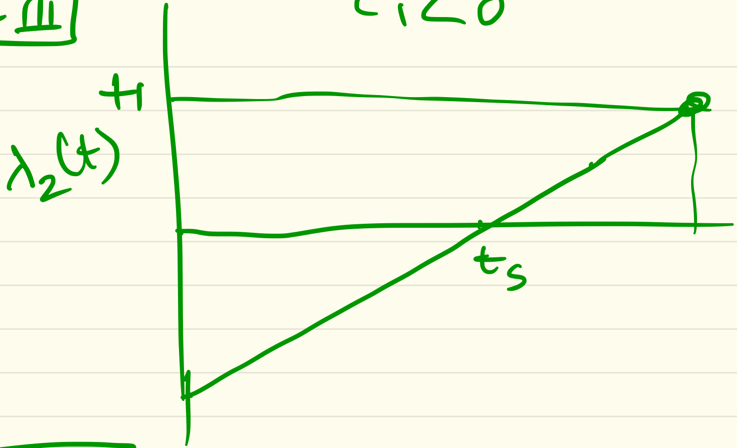
Case II

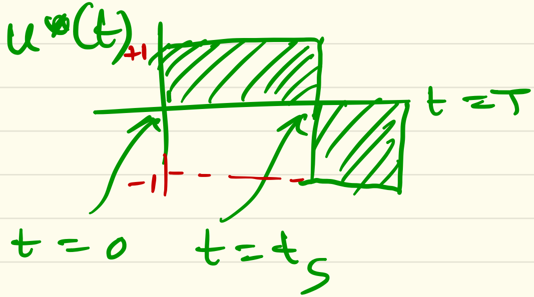




Case III

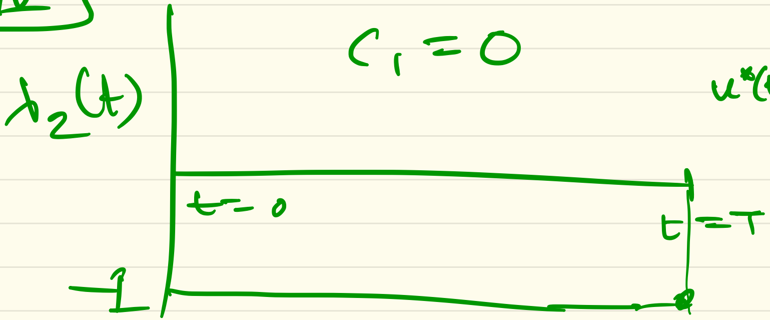
$c_1 < 0$

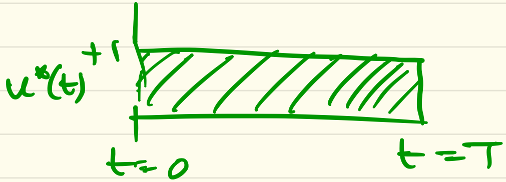




Case IV

$c_1 = 0$





Since $u^*(t) = \pm 1$, hence $\dot{x}_1 = x_2$

$\dot{x}_2 = \pm 1$

If $u^*(t) = +1$, then

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= +1 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} x_1(t) &= \tfrac{1}{2}t^2 + at + b \\ x_2(t) &= t + a \end{aligned}\right\}$$

a, b are constants of integration to be determined from the initial condition (I.C.)

Apply I.C. $\quad a = y_0, \; b = x_0$

$$t = x_2 - a$$

$$\therefore \; x_1 = \frac{1}{2}(x_2 - a)^2 + a(x_2 - a) + b$$

$x_1 \equiv x$

$x_2 \equiv y$

$$\Rightarrow x = \frac{1}{2}(y - y_0)^2 + y_0(y - y_0) + x_0$$

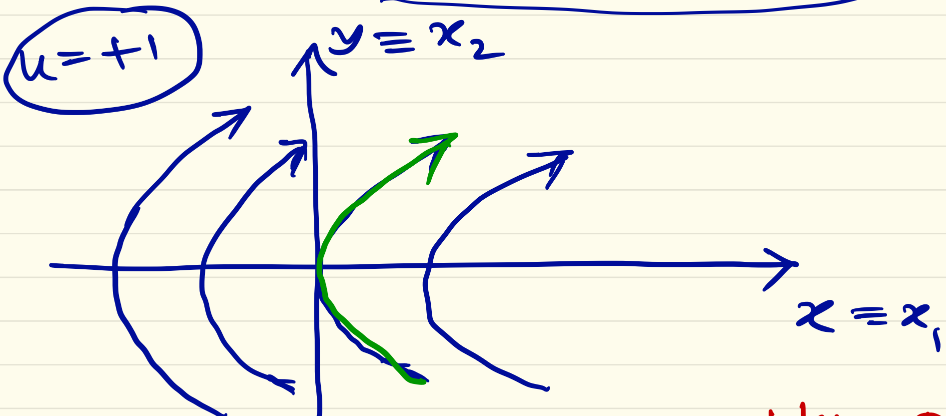
$$\Rightarrow (x - x_0) = \frac{1}{2}(y^2 - 2yy_0 + y_0^2) + yy_0 - y_0^2$$

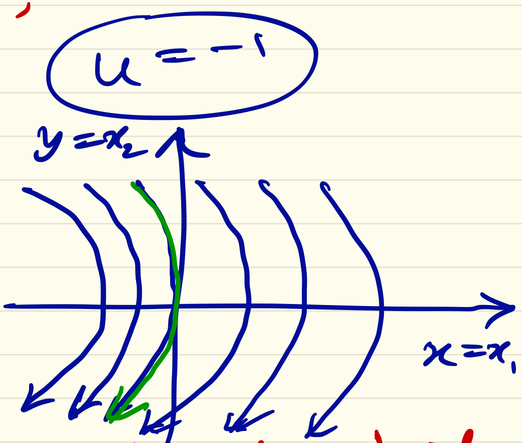
$$= \frac{1}{2}y^2 - \frac{1}{2}y_0^2 \Rightarrow \boxed{x = \frac{1}{2}y^2 + \left(x_0 - \frac{1}{2}y_0^2\right)}$$

$K \in \mathbb{R}$

Similarly, for $u = -1$, we get:

$$x = -\frac{1}{2}y^2 + K, \quad K \in \mathbb{R}.$$





Parameterized family of upgoing parabolas

$\dot{x}_1 = x_2$

$\dot{x}_2 = +1$

Parameterized family of downgoing parabolas

$\dot{x}_1 = x_2$

$\dot{x}_2 = -1$

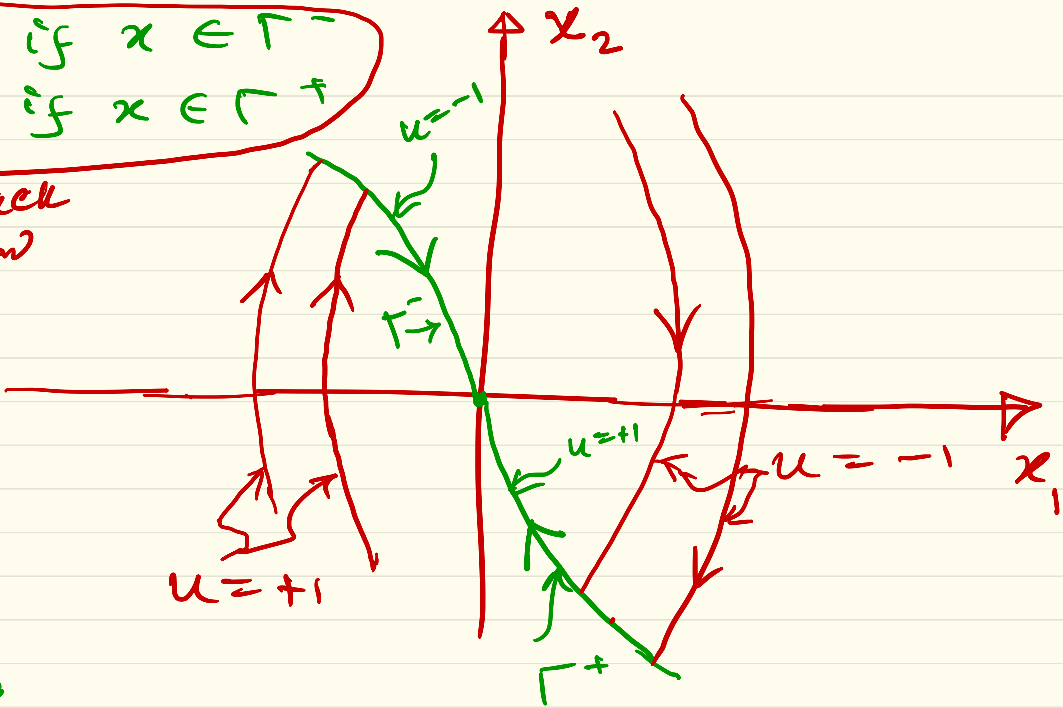
Only 2 of these curves pass through origin

Union of their segments define switching curve

$\Gamma \equiv \Gamma^+ \cup \Gamma^-$

$$u^* = \begin{cases} +1 & \text{if } x \in \Gamma^- \\ -1 & \text{if } x \in \Gamma^+ \end{cases}$$

State feedback control law



If the IC is already on Γ, then no switching

Optimal strategy : $u^* = +1$ or -1 depending on whether I.C. is above or below $\Gamma = \Gamma^+ \cup \Gamma^-$

Clearly, the switching curve $\Gamma = \Gamma^+ \cup \Gamma^-$ divides the state space in 2 regions, one above Γ, and other below Γ.

$$\left.\begin{array}{l} \text{If } x(t) \text{ is above } \Gamma, \text{ then } u^*(x(t)) = -1 \\ \text{If } x(t) \text{ is below } \Gamma, \text{ then } u^*(x(t)) = +1 \\ \text{If } x(t) \text{ is on } \Gamma^+, \text{ then } u^*(x(t)) = +1 \\ \text{If } x(t) \text{ is on } \Gamma^-, \text{ then } u^*(x(t)) = -1 \end{array}\right\}$$

End of example

The above example had dynamics:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \quad \begin{cases} \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \\ u \in \mathbb{R} \end{cases}$$

Similar results can be derived for $\underline{x} \in \mathbb{R}^n$, $\underline{u} \in \mathbb{R}^m$.

Of course, to bring the system to origin, we need (A, B) controllable.