Lecture #10

Infinite horizon LQR

Continuous Time

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q$$

Allow $T \to \infty$, $\phi \equiv 0 \Leftrightarrow M = 0$

$\dot{P} \equiv 0$.

CARE (Continuous-time Algebraic Riccati Eq^n.)

$$0 = A^T P_\infty + P_\infty A - P_\infty BR^{-1}B^T P_\infty + Q$$

Only make sense for LTI

Proposition (C1) (Existence, Uniqueness)

Let (A,B) be a controllable (actually, just need stabilizable) pair. Then $\exists !\, P_\infty \succcurlyeq 0$ that solves CARE.

Proposition (C2) $J^* = (x(0))^T P_\infty x(0)$

Discrete Time

$$P_K = Q + A^T P_{K+1}^{1/2}\left(I + P_{K+1}^{1/2} BR^{-1}B^T P_{K+1}^{1/2}\right)^{-1} P_{K+1}^{1/2} A$$

Allow, $N \to \infty$, $P_K = P_{K+1} =: P_\infty$, $M = 0$

DARE (Discrete-time Algebraic Riccati Eq^n) :

$$P_\infty = Q + A^T P_\infty^{1/2}\left(I + P_\infty^{1/2} BR^{-1}B^T P_\infty^{1/2}\right)^{-1} P_\infty^{1/2} A$$

Only makes sense for LTI

Proposition (D1) (Existence, Uniqueness)

ditto, i.e. $P_\infty \succcurlyeq 0$ solves DARE.

(D2) ditto (proof similar to finite horizon case done in class)

Proposition C3 (when is the closed-loop stable)

Suppose, $0 \preceq Q = C^T C$

Also, let (A, B) controllable (actually, need stabilizable)

and (A, C) observable, (actually, need detectable)

then

(i) the optimal closed-loop system $\dot{\underline{x}} = (A - BK_\infty)\underline{x}$

$$= (A - BR^{-1}B^T P_\infty)\underline{x}$$

is (asymptotically) stable

$\Leftrightarrow$ the matrix $(A - BK_\infty)$ is Hurwitz

$(\operatorname{Re}(\lambda_i) < 0)$

(ii) $P_\infty \succ 0$.

Proposition D3 (when is the closed-loop stable)

Suppose $0 \preceq Q = C^T C$.

Suppose (A, B) controllable and (A, C) observable,

Then $K_\infty = (R + B^T P_\infty B)^{-1} B^T P_\infty A$ makes the closed-loop system stable

$\Updownarrow$

$\underline{x}_{k+1} = (A - BK_\infty)\underline{x}_k$ is (asymp.) stable

$\Updownarrow$

The matrix $(A - BK_\infty)$ is Schur-Cohn stable

$(\max_i |\lambda_i| < 1)$

Proof of Proposition (C 3) (i) :

Let $\underbrace{A_{cl}}_{\text{closed-loop "A"}} := \left(A - BR^{-1}B^T P_\infty\right)$, and $\boxed{Q = CC^T}$

$\boxed{\text{Recall, } \underline{x} \in \mathbb{R}^n, \ \underline{u} \in \mathbb{R}^m}$

(This is always possible since $Q \succcurlyeq 0$
$\therefore$ By spectral decompos^n:
$$Q = VDV^{-1}$$
$$= VDV^T$$
$$= VD^{1/2}D^{1/2}V^T$$
$$\Rightarrow C = VD^{1/2})$$

Now by CARE, we have :

$$\boxed{A_{cl}^T P_\infty + P_\infty A_{cl} = -P_\infty BR^{-1}B^T P_\infty - CC^T}$$

Let $A_{cl}\underline{w} = \lambda \underline{w}$, $\underline{w} \neq 0$.

We would like to investigate under what condition, $\lambda \in \underbrace{\mathbb{C}^-}$, (open left half plane)

which is equivalent to closed-loop stability.

Let $(\lambda^*, \underline{w}^*)$ be the complex conjugate pair for $(\lambda, \underline{w})$

Pre-multiplying the above boxed eq^n by $\underline{w}^*$, and post-multiplying the same by $\underline{w}$, yields :

$$(\lambda + \lambda^*)\underline{w}^* P_\infty \underline{w} = -\underline{w}^* CC^T \underline{w} - \underline{w}^* P_\infty BR^{-1}B^T P_\infty \underline{w}$$

Since the RHS of the last eq^{n}: is the negative of a pos.-semidef. quadratic form, hence RHS ≤ 0.

On the LHS, $\underline{w}^* P_\infty \underline{w} \geq 0$ since $P_\infty \succcurlyeq 0$.

$\boxed{\text{Therefore, the only way LHS = RHS can happen is that } (\lambda+\lambda^*) \leq 0.}$

However, if $\lambda+\lambda^*=0$, then we get

$$0 = -\underline{w}^* C C^T \underline{w} - \underline{w}^* P_\infty B R^{-1} B^T P_\infty \underline{w}.$$

Since the RHS above is sum of two quadratics, hence $\lambda+\lambda^*=0$ mandates

$$C^T \underline{w} = \underline{0}_{n\times 1} \text{ and } R^{-1/2} B^T P_\infty \underline{w} = \underline{0}_{m\times 1}$$

$\Downarrow$ ($\because A_{cl} := A - BR^{-1}B^T P_\infty$)

$$A\underline{w} = A_{cl}\underline{w} = \lambda \underline{w}.$$

$$\Leftrightarrow \boxed{C^T \underline{w} = 0_{n\times 1} \text{ and } A\underline{w} = \lambda \underline{w}}$$

Recall from linear systems theory that
(A, C) detectable $\Longleftrightarrow$ $A\underline{w} = \lambda \underline{w}$, $C^T \underline{w} = 0$, $\underline{w} \neq 0$
implies $Re(\lambda) < 0$.

However, $Re(\lambda) = \frac{\lambda + \lambda^*}{2} = 0$ in this case, which is a contradiction. Therefore, we cannot have
$\lambda + \lambda^* = 0$.

At this point, we know that $\lambda + \lambda^* \leq 0$ and that $\lambda + \lambda^* \neq 0$.

$$\therefore \lambda + \lambda^* < 0 \Leftrightarrow Re(\lambda) < 0.$$
$$\Updownarrow$$
A_{cl} is Hurwitz. $\blacksquare$

This proof is from Anderson & Moore, Optimal Control: Linear Quadratic Methods, ch 3.2

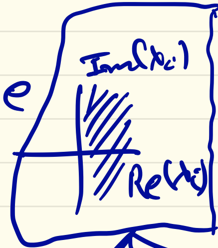
Stabilizable (in continuous-time) — Some Background info

$\Updownarrow$

Given (A, B), there exist some matrix K, s.t. $(A - BK)$ is Hurwitz, i.e., $Re(\lambda_i(A-BK)) < 0 \ \forall i$.

$\Updownarrow$

uncontrollable subspace is naturally stable (look up Kalman decomposition)



$\boxed{\text{Theorem}}$ The pair (A, B) is

- controllable iff $\text{rank}\left[(\lambda I - A) \mid B\right] = n \quad \forall \lambda \in \mathbb{C}$.
- stabilizable iff $\text{rank}\left[(\lambda I - A) \mid B\right] = n \quad \forall \lambda \in \mathbb{C}^+$

Here A is $n \times n$
B is $n \times m$, $m \neq n$.

MATLAB command to compute LQR infinite horizon

(Computing P_∞)

$$\gg [K_\infty, P_\infty, e] = lqr(sys, \underbrace{Q, R, S}_{\text{matrices appearing in the cost function}})$$

K_∞: Kalman gain

P_∞: solⁿ of CARE or DARE

e: eigenvalues of the closed-loop matrix $(A - BK_\infty)$

$$\gg sys = ss(A, B, \cancel{C}, \cancel{D}) \longleftrightarrow \dot{x} = Ax + Bu$$

($C \to 0$, $D \to 0$)

>> sys.A (will print the A matrix)

(You could create "sys" as discrete time system by passing extra argument (T_s : sampling time), e.g. sys = ss(A,B,C,D,T_s))

Handling Additional Constraints in the OCP (continuous time)

#1 Integral / Isoperimetric constraints:

$$\int_0^T N(\underline{x}, \underline{u}, t)\,dt \quad \text{must be conserved.}$$

$\underline{x} \in \mathbb{R}^n$

(i.e.) $$\int_0^T N(\underline{x}, \underline{u}, t)\,dt = K$$ ← given

To handle this: introduce extra state x_{n+1}

s.t. $$\dot{x}_{n+1} = N(\underline{x}, \underline{u}, t)$$

and $\underline{x_{n+1}(0) = 0}$, $\underline{x_{n+1}(T) = K}$ (given)

Now apply necessary conditions to Hamiltonian:

$$H = L + \underbrace{\lambda^T}_{n \times 1} \underbrace{f}_{n \times 1} + \lambda_{n+1}(t)\, N$$

In particular, $$\dot{\lambda}_{n+1} = -\frac{\partial H}{\partial x_{n+1}} = 0 \iff \boxed{\lambda_{n+1} = \text{Constant}}$$

#2 Control Equality constraint:

$$c(\underline{u}, t) = 0, \quad \underline{u} \in \mathbb{R}^m, \quad m \geq 2.$$

scalar function

(For $m=1$, this doesn't make sense since then there is no OCP to solve)

Augment the Hamiltonian: $H = L + \lambda^T f + \mu(t) c$

PMP

$$\underline{0} = \frac{\partial H}{\partial \underline{u}} = \frac{\partial L}{\partial \underline{u}} + (\lambda(t))^T \frac{\partial f}{\partial \underline{u}} + \mu(t) \frac{\partial c}{\partial \underline{u}}.$$

#3 Equality constraints on f^ns of control and state:

$$c(\underline{x}, \underline{u}, t) = 0 \quad \text{where} \quad \frac{\partial c}{\partial \underline{u}} \neq 0 \quad \text{for any } \underline{u}.$$

Again,

$$H = L + \lambda^T f + \mu(t) c$$

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} = -\frac{\partial L}{\partial \underline{x}} - \lambda^T(t) \frac{\partial f}{\partial \underline{x}} - \underline{\mu}(t) \frac{\partial c}{\partial \underline{x}}.$$

#4 Equality constraint on functions of state variables:

$$S(\underline{x}, t) = 0 \quad \text{---} (*) \quad \text{for all } t_0 \le t \le T$$

$$\frac{d}{dt} S = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} \cancel{\dot{x}}^{\,f(\underline{x},u,t)} = 0 \quad \text{---} (**)$$

Now, (**) may or may not have explicit dependence on $\underline{u}$.

→ If it does, then (**) plays the role of $C(\underline{x}, \underline{u}, t) = 0$ as in #3

However, we must either eliminate one component of $\underline{x}$ in terms of the remaining $(n-1)$ components using (*). OR add (*) as a B.C. @ $t = t_0$ or $t = T$.

→ If (**) still does not have explicit dependence on u, then do d/dt again

↓

keep doing until u appears explicitly

Suppose this happens @ q^{th} order $\frac{d}{dt}$

Then $\underbrace{S^{(q)}(\underline{x}, \underline{u}, t) = 0}_{\text{plays the role of } c(\underline{x},\underline{u},t)=0.}$ where $S^{(q)} = \frac{d^q S}{dt^q}$.

In addition, we must eliminate q components of $\underline{x}$ in terms of the remaining $(n-q)$ components using q algebraic eq$\overset{=}{\text{n}}$s:

$$\begin{pmatrix} S(\underline{x},t) \\ S^{(1)}(\underline{x},t) \\ \vdots \\ S^{(q-1)}(\underline{x},t) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{(q-1)\times 1}.$$