Lecture #1

# of agents/ Decision makers/ problem solvers	Static	Dynamic
1	Optimization problem (OPT) →	Optimal Control Problem (OCP)
> 1	Game (Theory) → Co-operative, Non-cooperative	Differential Game

Development

Finite dimensional decision making
(decision variable = scalar, vector, matrix)

Optimization (OPT)

↓

Infinite dimensional Optimization
(Decision variable = function)

Calculus of Variations (CoV)

↓

Optimal Control Problem (OCP)

Has Differential equation as constraint

Development

ancient times

Optimization (OPT)

↓

early encounters (< 10 BC)

Beginning (late 17th century)

Calculus of Variations (CoV)

1687, 1696 ....

1700s (Euler-Lagrange)

↓

Optimal Control Problem (OCP)

1950s - 1960s

Soviet Union (L.V. Pontryagin)

US (Richard Bellman)

Overview

$$\min_{\boldsymbol{x} \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(\boldsymbol{x})$$

$$\downarrow$$

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right) \, \mathrm{d}\boldsymbol{x}$$

Overview

$f : \mathbb{R}^n \mapsto \mathbb{R}$

Decision variable $= x$

$$\min_{x \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(\boldsymbol{x})$$

(Optimization/ OPT)

function

(Calculus of Variations/CoV)

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x}$$

Decision variable $= f(x)$

Functional

($f^{\underline{n}}$ of a $f^{\underline{n}}$)

$$\min_{u(\cdot) \in \mathcal{U}([0,T]) \subseteq \mathcal{F}([0,T])} \quad J(\boldsymbol{u})$$

Decision variable $u(\cdot)$

(Optimal Control Problem/ OCP)

$$\text{subject to} \quad \dot{z}(t) = \boldsymbol{\phi}\left(z(t), \boldsymbol{u}(t), t\right)$$

OPT example: Least squares

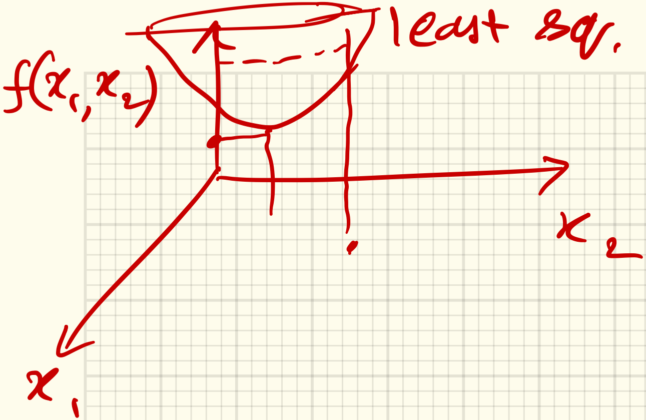
OPT template: $\min_{x \in \mathcal{S} \subseteq \mathbb{R}^n} f(x)$

$\parallel$

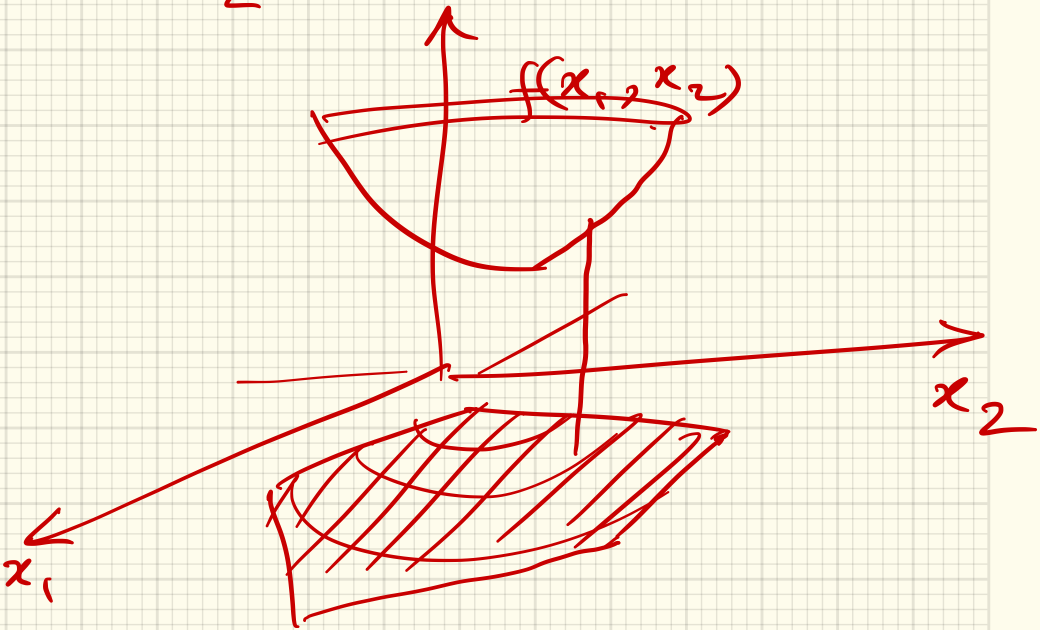
In this problem: $\min_{x} \| Ax - b \|_2^2$

unconstrained least sq. : $\mathcal{S} = \mathbb{R}^n$

Constrained least sq. $\mathcal{S} \subset \mathbb{R}^n$ } $\mathcal{S} : A\underline{x} \leq \underline{b}$



Constrained least square



OPT example: Least squares

OPT template: $\min_{\boldsymbol{x} \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(\boldsymbol{x})$

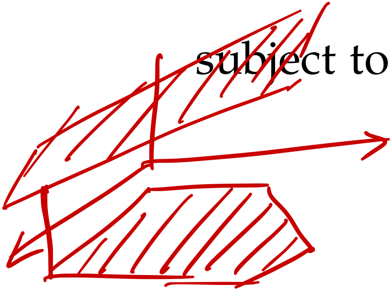
In this problem: $\min_{\boldsymbol{x}} \quad \| A\boldsymbol{x} - \boldsymbol{b} \|_2^2$

$$\mathcal{S} = \mathbb{R}^n, \qquad f(\boldsymbol{x}) = \| A\boldsymbol{x} - \boldsymbol{b} \|_2^2$$

OPT example: two variable LP

OPT template: $\displaystyle \min_{x \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(x)$

In this problem: $\displaystyle \max_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2} \quad 15x_1 + 10x_2$



$$\text{subject to} \quad \frac{1}{4}x_1 + x_2 \leq 65,$$

$$\frac{5}{4}x_1 + \frac{1}{2}x_2 \leq 90,$$

$$x_1, x_2 \geq 0$$

OPT example: two variable LP

OPT template: $$\min_{\boldsymbol{x} \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(\boldsymbol{x})$$

In this problem: $$\max_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2} \quad 15x_1 + 10x_2$$

$$\text{subject to} \quad \frac{1}{4}x_1 + x_2 \leq 65,$$

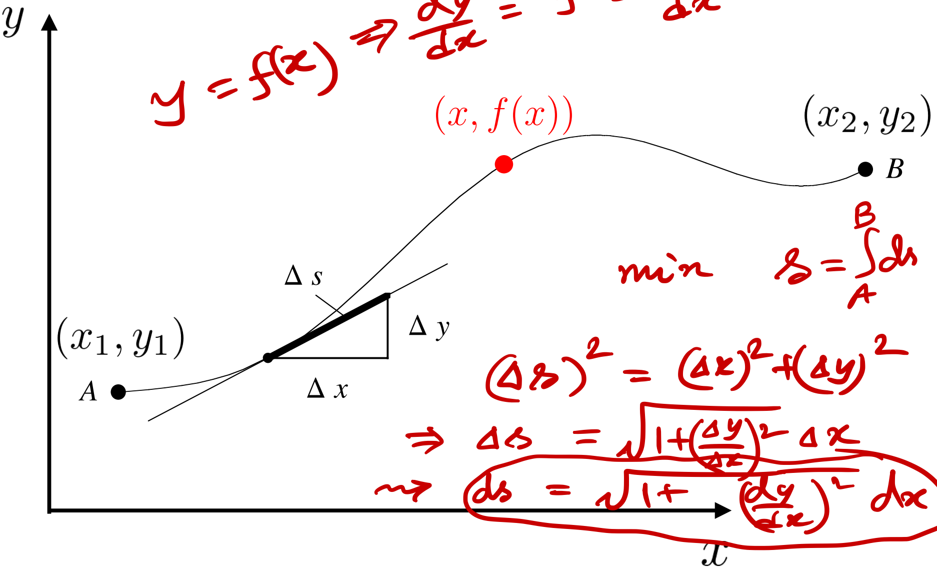
$$\frac{5}{4}x_1 + \frac{1}{2}x_2 \leq 90,$$

$$x_1, x_2 \geq 0$$

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^2 \,:\, A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\} \subset \mathbb{R}^2$$

CoV example: Shortest planar path

$$y = f(x) \Rightarrow \frac{dy}{dx} = f' \equiv \frac{df}{dx}$$



$$\min \quad s = \int_A^B ds$$

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Rightarrow \Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

$$\rightsquigarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

CoV example: Shortest planar path

CoV template:

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right) \, \mathrm{d}\boldsymbol{x}$$

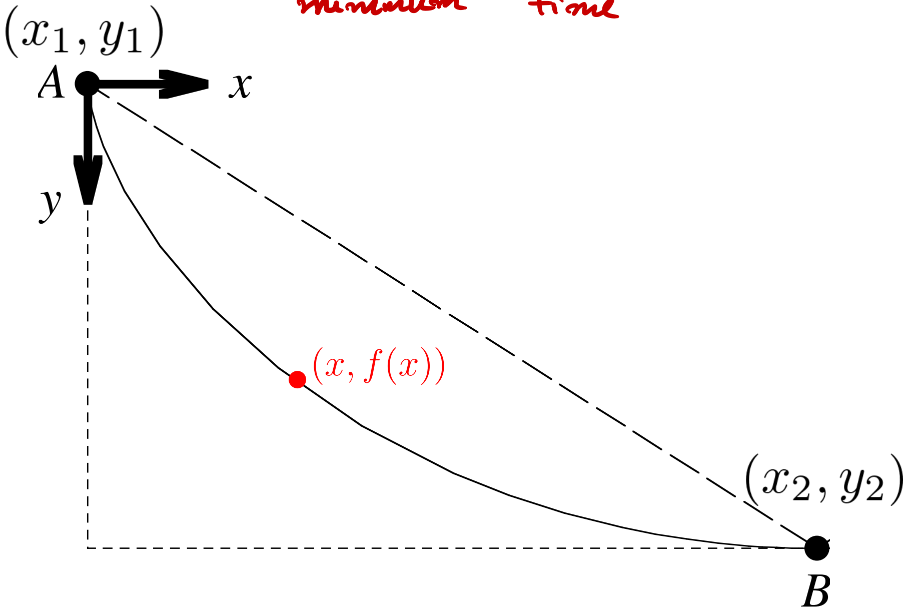
In this problem:

$$I(f) = \int_{x_1}^{x_2} \sqrt{1 + (f')^2} \, \mathrm{d}x$$

$$\mathrm{dom}(f) = [x_1, x_2], \text{ assuming } x_1 \neq x_2$$

$$\mathcal{F}(\mathbb{R}) = \{f \in C^1(\mathbb{R}) \, : \, f(x_1) = y_1, f(x_2) = y_2\}$$

CoV example: $\underbrace{\text{Brachisto}}_{\text{minimum}}\underbrace{\text{chrone}}_{\text{time}}$ (1696)



CoV example: Brachistochrone (1696)

CoV template:

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right) \, \mathrm{d}\boldsymbol{x}$$

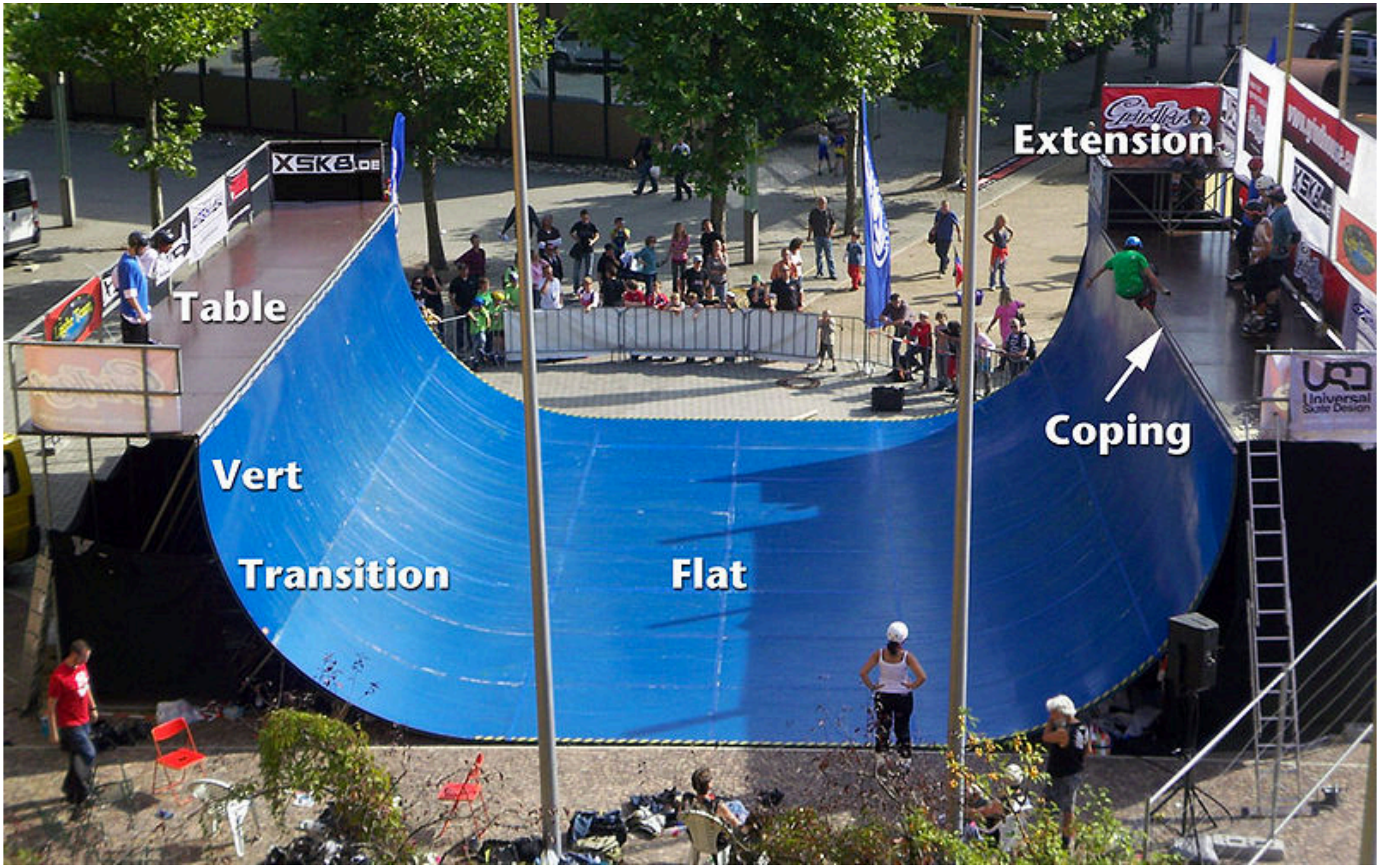
In this problem:

$$I(f) = \int_{x_1}^{x_2} \sqrt{\frac{1 + \left(f'\right)^2}{f}} \, \mathrm{d}x$$

$$\mathrm{dom}(f) = [x_1, x_2], \, x_1 \neq x_2, \, y_1 > y_2$$

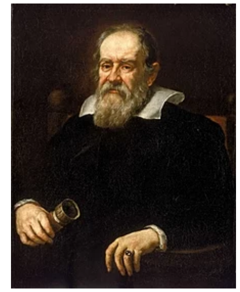
$$\mathcal{F}(\mathbb{R}) = \{f \in C^1\left(\mathbb{R}\right) \, : f\left(x_1\right) = y_1, f\left(x_2\right) = y_2\}$$

Optimal shape of Skateboard Ramp



June 1696 Challenge in Acta Eruditorum Journal

Johann Bernoulli
(Posted problem in 1696)



Galileo Galilei
(Conjecture in 1638)

6 Solutions Appeared in May 1697 Issue

Jakob Bernoulli

Gottfried Leibniz

Guillaume de l'hôpital

Ehrenfried Tschirnhaus

Mr. Anonymous

"ex unge leonem" — Johann Bernoulli

Conditions for optimality

OPT	CoV	OCP
(unconstrained) $\frac{\partial f}{\partial x} = 0$	?	?
(constrained) $\frac{\partial}{\partial x}(f + \underline{\lambda}^T \underline{g}) = 0$ KKT conditions	?	?

CoV theory: Integral constraints

CoV template:

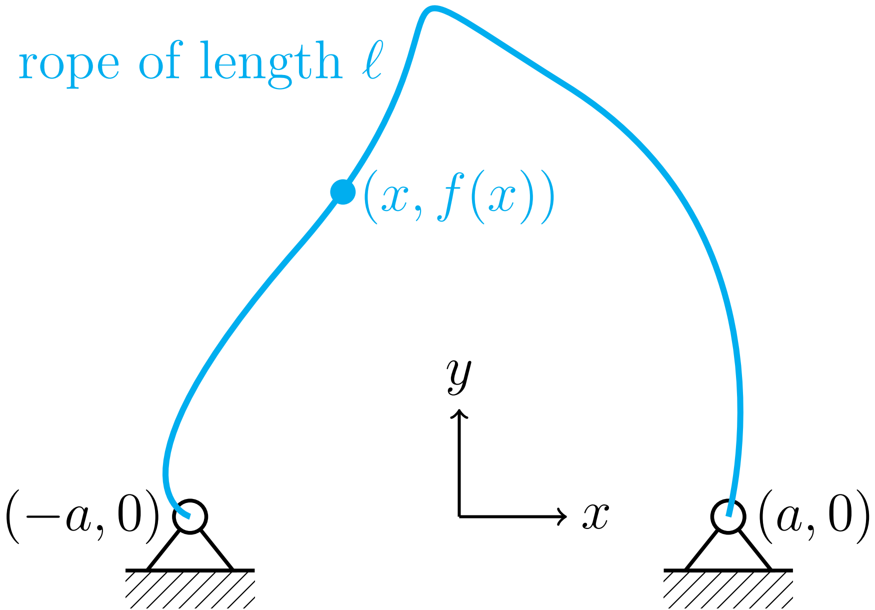
$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x}$$

$$\text{subject to} \int_{\mathrm{dom}(f)} \boldsymbol{M}\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x} = \boldsymbol{k}$$

Euler-Lagrange equation:

$$\frac{\partial}{\partial f}\left(L + \boldsymbol{\lambda}^\top \boldsymbol{M}\right) - \nabla \cdot \frac{\partial}{\partial \nabla f}\left(L + \boldsymbol{\lambda}^\top \boldsymbol{M}\right) = 0$$

CoV example: Isoperimetric problem

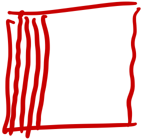


CoV example: Isoperimetric problem

CoV template:

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x}$$

$$\text{subject to} \int_{\mathrm{dom}(f)} \boldsymbol{M}\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x} = \boldsymbol{k}$$



In this problem:

$$\text{minimize } I(f) = \int_{-a}^{+a} f\left(x\right)\, \mathrm{d}x,\ 0 < 2a < \ell, \text{ subject to}$$

$$\int_{-a}^{+a} \sqrt{1 + (f')^2}\, \mathrm{d}x = \ell \text{ (given)}, f(-a) = f(a) = 0$$