AMS 232: Applied Optimal Control

Spring 2019

Midterm Exam

Name:____ Student ID:______________________________

For this exam you only need a pen/pencil. No electronic device is allowed. Write your answers in the spaces provided. If you need more space, work on the other side of the page.

Problem 1 (5 + 5 + 8 + 12 + (5 + 5) = 40 points)

Suppose the (scalar) state $x(t)$ of national economy is governed by the 2nd order ODE

$$\ddot{x} = -\alpha^2 x + u, \quad \alpha \in \mathbb{R}, \quad t \geq 0, \quad x(0) = \dot{x}(0) = 0,$$

where $u(t)$ is the effort the Government puts at time t for economic reform. The Government would like to maximize its chance of getting re-elected at the (fixed) terminal time T, by bringing the national economy at a healthy state at the time of re-election, while not spending too much effort in economic reform during its tenure, i.e.,

$$\underset{u(\cdot)}{\text{maximize}} \quad x(T) - \int_0^T u^2 \, \mathrm{d}t.$$

In practice, the Government may want to maximize an increasing function of the same, but we will consider the above cost function here.

(a) Define the state vector, and then write the 2nd order ODE in state space form, i.e., as a controlled vector first order ODE.

The state vector is $\boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in \mathbb{R}^2$. In our case, $u \in \mathbb{R}$. Therefore, the controlled dynamics in state space form is

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

(b) Using your answer in part (a), clearly re-write the optimal control problem in standard form. Identify terminal constraint and/or terminal cost, if any.

The optimal control problem in standard form is

$$\underset{u(\cdot)}{\text{minimize}} \quad -x_1(T) + \int_0^T u^2 \mathrm{d}t$$

$$\text{subject to}\quad \dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

The final time T is fixed; there is no terminal constraint ($\psi \equiv 0$); the terminal cost $\phi = -x_1(T)$.

(c) Write the Hamiltonian, the costate ODEs, the PMP, and the transversality condition for the optimal control problem in part (b).

The Hamiltonian: $H = u^2 + \lambda_1 x_2 + \lambda_2 \left(-\alpha^2 x_1 + u\right)$.

The costate ODEs:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = \alpha^2 \lambda_2, \quad \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1.$$

PMP: $0 = \dfrac{\partial H}{\partial u} = 2u + \lambda_2 \quad \Rightarrow \quad u = -\dfrac{1}{2}\lambda_2.$

Transversality: Since $\mathrm{d}T = 0$, $\mathrm{d}\boldsymbol{x}(T) \neq \boldsymbol{0}$, hence $\dfrac{\partial \phi}{\partial \boldsymbol{x}(T)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1(T) \\ \lambda_2(T) \end{pmatrix}.$

(d) Find the costates in terms of t, α, T. (Hint: Use the transversality conditions to solve the costate ODE initial value problem.)

From the costate ODEs in part (c), we have

$$\ddot{\lambda}_1 = \alpha^2 \dot{\lambda}_2 = -\alpha^2 \lambda_1 \quad \Rightarrow \quad \ddot{\lambda}_1 + \alpha^2 \lambda_1 = 0 \quad \Rightarrow \quad \lambda_1(t) = a\cos(\alpha t) + b\sin(\alpha t),$$

where a, b are constants to be determined from the terminal values of the costates. Consequently, $\lambda_2(t) = \dfrac{1}{\alpha^2}\dot{\lambda}_1 = \dfrac{1}{\alpha}\left(-a\sin(\alpha t) + b\cos(\alpha t)\right)$.

To determine the constants a, b, we now use the terminal values of the costates (coming from transversality): $\lambda_1(T) = -1$, $\lambda_2(T) = 0$. This gives:

$$\begin{pmatrix} \cos(\alpha T) & \sin(\alpha T) \\ -\frac{1}{\alpha}\sin(\alpha T) & \frac{1}{\alpha}\cos(\alpha T) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\alpha T) & \sin(\alpha T) \\ -\frac{1}{\alpha}\sin(\alpha T) & \frac{1}{\alpha}\cos(\alpha T) \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\frac{1}{\alpha}\left(\cos^2(\alpha T) + \sin^2(\alpha T)\right)} \begin{pmatrix} \frac{1}{\alpha}\cos(\alpha T) & -\sin(\alpha T) \\ \frac{1}{\alpha}\sin(\alpha T) & \cos(\alpha T) \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos(\alpha T) \\ -\sin(\alpha T) \end{pmatrix}.$$

(e) Find the optimal economic reform policy $u^*(t)$ for the Government. Also find the terminal reform $u^*(T)$.

Combining the PMP and part (d), we get

$$u^*(t) = -\frac{1}{2}\lambda_2^*(t) = -\frac{1}{2\alpha}\left(-a\sin(\alpha t) + b\cos(\alpha t)\right) = \frac{1}{2\alpha}\left(\sin(\alpha T)\cos(\alpha t) - \cos(\alpha T)\sin(\alpha t)\right),$$

i.e., $u^*(t) = \dfrac{1}{2\alpha}\sin\left(\alpha(T-t)\right)$.

Therefore, the terminal reform $u^*(T) = \dfrac{1}{2\alpha}\sin 0 = 0$, assuming $\alpha \neq 0$.

Problem 2 (5 × 2 = 10 points)

For each the following statements, ONLY ONE among the three options are correct. Choose the correct option for each. You DO NOT need to provide any explanation.

1. Denote the solution of Riccati initial value problem in a finite horizon LQR at $t = 0$ as $\boldsymbol{P}(0)$. Then the worst-case initial condition $\boldsymbol{x}_0$ on the unit sphere $\boldsymbol{x}_0^\top \boldsymbol{x}_0 = 1$, i.e., the unit vector that maximizes the minimum value of the cost, is

   (i) the eigenvector $\boldsymbol{v}_{\min}$ associated with $\lambda_{\min}\left(\boldsymbol{P}(0)\right)$.

   (ii) the eigenvector $\boldsymbol{v}_{\max}$ associated with $\lambda_{\max}\left(\boldsymbol{P}(0)\right)$.

   (iii) independent of $\boldsymbol{P}(0)$.

   Answer: (ii)

2. Consider a discrete time finite horizon LQR problem:

$$\underset{\{\boldsymbol{u}_t\}_{t=0}^{N-1}}{\text{minimize}} \quad \frac{1}{2}\left\{\boldsymbol{x}_N^\top \boldsymbol{M}\boldsymbol{x}_N + \sum_{t=0}^{N-1}\left(\boldsymbol{x}_t^\top \boldsymbol{Q}\boldsymbol{x}_t + \boldsymbol{u}_t^\top \boldsymbol{R}\boldsymbol{u}_t\right)\right\}$$

$$\text{subject to} \quad \boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{u}_t, \quad t = 0, 1, \ldots, N-1.$$

   with fixed terminal time $t = N$. The matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{M})$ are constants with standard assumptions $\boldsymbol{M}, \boldsymbol{Q} \succeq \boldsymbol{0}$, $\boldsymbol{R} \succ \boldsymbol{0}$. The pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. Then the Kalman gain matrix $\boldsymbol{K}_t$ defining the optimal state feedback at time t

   (i) is always a constant matrix.

   (ii) is always a time varying matrix.

   (iii) may or may not be time varying depending on the matrix $\boldsymbol{M}$.

   Answer: (iii)

3. Suppose we would like to solve a continuous time optimal control problem in Mayer form, where the terminal cost $\phi(\cdot)$ is independent of the terminal time T, but only depends on the terminal state $\boldsymbol{x}(T)$, i.e., $\phi \equiv \phi(\boldsymbol{x}(T))$. Then the Hamiltonian H^* along the optimal trajectory

   (i) must remain constant in time.

   (ii) must be time varying.

(iii) may or may not be constant in time.

Answer: (iii)

4. Let $\Omega \subset \mathbb{R}$, and $u : \Omega \mapsto \mathbb{R}$. The Euler-Lagrange equation for the calculus of variations problem: $\underset{u \in C^2(\Omega)}{\text{minimize}} \int_\Omega L(x, u, u', u'')\, \mathrm{d}x$, is given by

   (i) $L_u = (L_{u'})'$.
   (ii) $L_u = (L_{u'})' + (L_{u''})''$.
   (iii) $L_u = (L_{u'})' - (L_{u''})''$.

   Answer: (iii)

5. In a continuous time finite dimensional optimal control problem, the dimension of the costate vector

   (i) equals the dimension of the control vector.

   (ii) equals the dimension of the state vector.

   (iii) equals unity.

   Answer: (ii)

Some useful information

- A standard finite dimensional optimal control problem in continuous time reads

$$\begin{aligned}
\underset{\boldsymbol{u}(\cdot)}{\text{minimize}} \quad & \phi\left(\boldsymbol{x}(T), T\right) + \int_0^T L\left(\boldsymbol{x}, \boldsymbol{u}, t\right)\, \mathrm{d}t \\
\text{subject to} \quad & \dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, t\right), \quad \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{u} \in \mathbb{R}^m, \\
& \boldsymbol{\psi}\left(\boldsymbol{x}(T), T\right) = \boldsymbol{0}.
\end{aligned}$$

- The Hamiltonian $H := L + \langle \boldsymbol{\lambda}, \boldsymbol{f} \rangle$, where $\boldsymbol{\lambda}(t)$ is the costate vector at time t.

- The first order necessary conditions for optimality are:

$$\text{State ODE:} \quad \dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, t\right), \quad \text{Costate ODE:} \quad \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}}, \quad \text{PMP:} \quad \boldsymbol{0} = \frac{\partial H}{\partial \boldsymbol{u}},$$

$$\text{Transversality:} \quad \left(\frac{\partial \phi}{\partial \boldsymbol{x}} + \left(\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}}\right)^\top \boldsymbol{\nu} - \boldsymbol{\lambda}\right)^\top \Bigg|_{t=T} \mathrm{d}\boldsymbol{x}(T) + \left(\frac{\partial \phi}{\partial t} + \left(\frac{\partial \boldsymbol{\psi}}{\partial t}\right)^\top \boldsymbol{\nu} + H\right) \Bigg|_{t=T} \mathrm{d}T = 0.$$